UNIVERSITY OF WARSAW

On the fractal reconstruction of velocity and scalar fields in turbulent flow

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Atmospheric physics seminar, 01.04.2022

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What is a fractal?

A fractal is a pattern that repeats itself at different scales. This property is called „Self-Similarity”.

Figure: Romanesco cauliflower (Pixabay public domain picture)

Examples of fractals

Fractals are found all over nature...

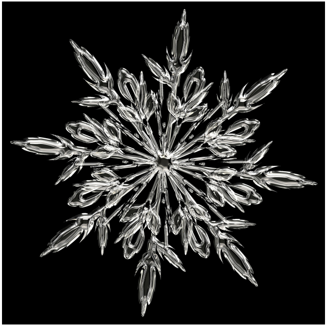


Figure: A snowflake (Pixabay public domain photo)

Figure: Trees (Pixabay public domain photo)

Examples of fractals

... in geometry fractals can be created by repeating a simple process...

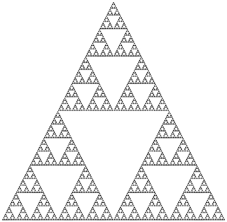
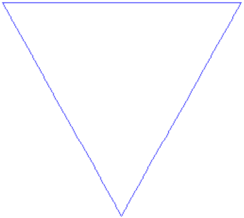


Figure: Sierpiński triangle (public domain).



Von Koch snowflake (public domain, author: António Miguel de Campos)

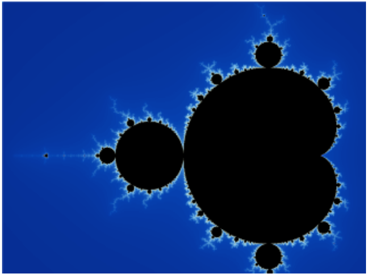
Examples of fractals

...in algebra - by calculating simple nonlinear equations over and over again.

Mandelbrot set - a set of complex numbers C for which the function

$$f(z) = z^2 + C$$

does not diverge to infinity when iterated from $z = 0$.



Mandelbrot zoom animation (public domain)

Fractal dimension

Calculating length of the Koch curve with sticks.

$$L_0 = 1$$

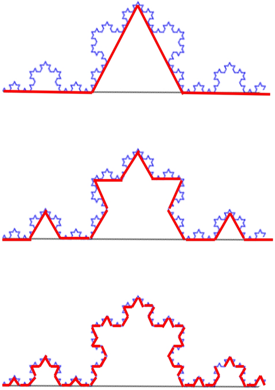
$$L_1 = 4 \cdot \frac{1}{3}$$

$$L_2 = 4^2 \cdot \frac{1}{3^2}$$

$$L_3 = 4^3 \cdot \frac{1}{3^3}$$

$$\dots$$

$$L_n = 4^n \cdot \left(\frac{1}{3}\right)^n$$



Fractal dimension

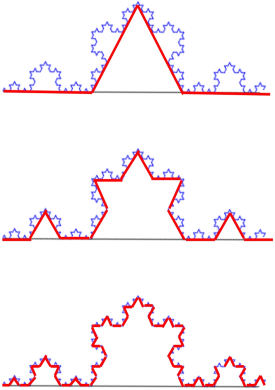
Calculating lengths of the Koch curve with sticks.

$$L_n = 4^n \cdot \left(\frac{1}{3}\right)^n = \left(\frac{4}{3}\right)^n$$

For $n \to \infty$, $L_n \to \infty$
Consider

$$N = \epsilon^{-D}$$

where
N - number of offsprings,
ϵ - rescaling factor,
D - dimension.



Fractal dimension

Calculating lengths of the Koch curve with sticks.

$$N = \epsilon^{-D}$$

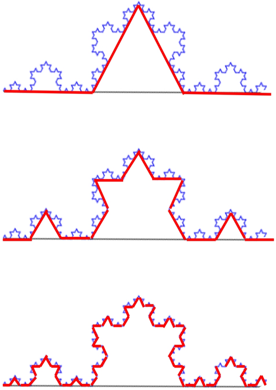
For non-fractal curves

$$3 = \left(\frac{1}{3}\right)^{-D}, \quad D = 1,$$

where D is the dimension

For the von Koch curve

$$4 = \left(\frac{1}{3}\right)^{-D}, \quad D = 1.2619,$$



Fractals...

- ...are pretty
- ...are patterns that repeat themselves at different scales
- ...can be found in nature (snowflakes, trees, spirals)
- ...can be created by repeating a simple process
- ...or iterative calculations with the use of nonlinear equations
- ...have non-integer dimensions.

Fractals and turbulence

Coastline paradox.

Lewis Fry Richardson tried to measure the coastline of the Great Britain with a ruler...
... and failed
(Because the coastline is a fractal structure.)

Fractals and turbulence

Eddy cascade.

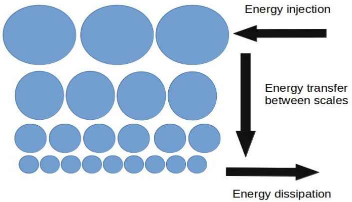


Figure: Richardson-Kolmogorov's cascade picture

Lewis Fry Richardson was the author of the famous poem about turbulence:

Big whirls have little whirls that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity,

Richardson-Kolmogorov's cascade

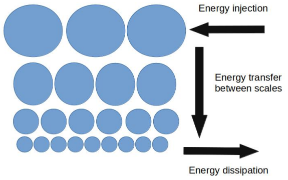


Figure: Energy cascade

- energy is injected at large scales (shear production, buoyancy production)
- .... it is transported towards smaller and smaller scales in the energy cascade
- ... it is converted into heat at the smallest scales $\mathcal{O}(0.001m)$–$\mathcal{O}(0.01m)$

Kolmogorov’s similarity hypothesis

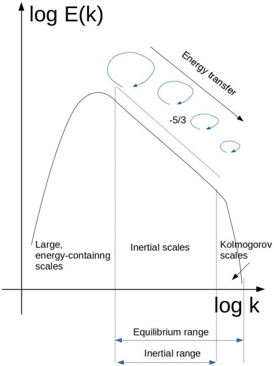


Figure: Energy spectrum

Assumes that scale similarity exist within the inertial range scales

Energy spectrum

$$E(\kappa) \sim \epsilon^{2/3} \kappa^{-5/3}$$

κ - wavenumber,
ϵ - dissipation rate

Structure function

$$S(r) \sim (\epsilon r)^{2/3}$$

κ - wavenumber,

Intermittency - β model of Frish et al.

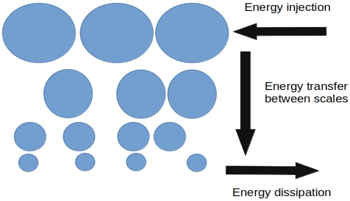


Figure: Intermittent energy cascade

- Frisch et al. [J. Fluid Mech., **87**, 1978] proposed the β model
- They assumed that after n generations only a fraction of the space β^n is occupied by an active fluid and

$$N = \left(\frac{1}{2}\right)^{-D}$$

where $D \approx 2.5$

- This gives an intermittency correction to the structure functions

Fractal structure of clouds

Investigation of the cloud-clear air interface

Malinowski & Zawadzki [JAS, **50**, 1993]

The estimated fractal dimension of the cloud-clear air interface is

$$D = 2.55$$

Figure: Cloud (Pixaby, public domain)

2D turbulence and SLE curves

Schramm-Loewner SLE curves with a parameter κ are random curves of the fractal dimension

$$D = 1 + \frac{\kappa}{8}$$

They are the scaling limit of a variety of two-dimensional lattice models.

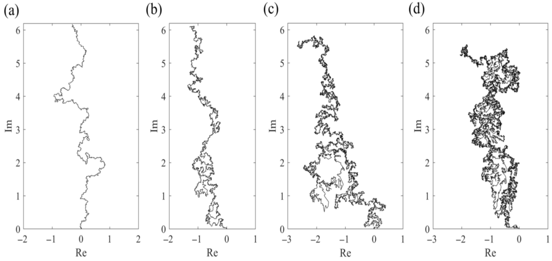


Figure: SLE curves, Shibasaki & Saito [Entropy, **23**, 2021], CC-BY 4.0

2D turbulence and SLE curves

Schramm-Loewner SLE curves are also boundaries of large vorticity clusters in 2D turbulence. [Bernard et al. Nature, **2**, 2006].

SLE curves are conformally invariant curves.

Probability measure of zero-vorticity in 2D inviscid turbulence is conformally invariant [Wacławczyk et al., Phys. Rev. Fluids, **6**, 2021]

Figure: 2D turbulence in Jupiter's atmosphere (NASA, public domain)

Fractal Interpolation technique (FIT)

Turbulent velocity signals have a fractal dimension

$$D \approx 1.7,$$

which is close to the value

$$D = 5/3 \approx 1.6667$$

which is expected for Gaussian processes with a $-5/3$ spectrum
[Scotti et al. Phys. Rev. E **51**, 1995]

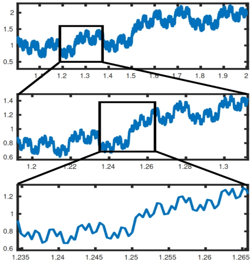


Figure: Fractal signal.

Fractal Interpolation technique (FIT)

The FIT is an iterative mapping procedure to construct synthetic small-scale structures of any field (e.g velocity) from the knowledge of its filtered field. [Scotti & Meneveau, Physica D, 1999]

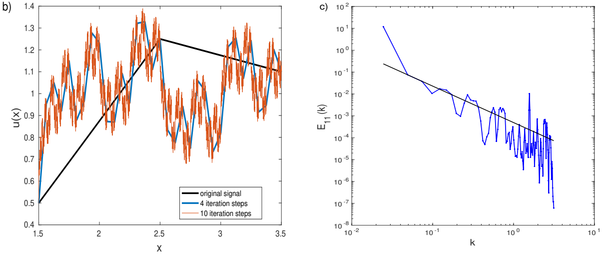
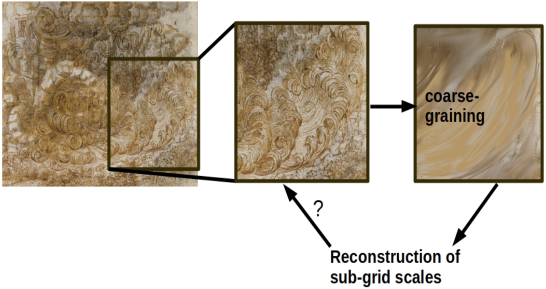


Figure: a) Different stages during the construction of a fractal function after 0,1 and 10 iterations b) Energy spectrum of the constructed signal.

Fractal reconstruction

Large Eddy Simulation method introduces spatial filtering (i.e. effective coarse-graining of turbulent field).

FIT can be used as a subgrid-scale model which allows to reproduce and explain roboust characteristics of the sub-grid scales.



Fractals and turbulence

- Richardson's energy cascade picture – self-similarity of scales
- Phenomenon of intermittency – explained by the fractal structure
- Fractal structure of surfaces in turbulence (e.g. cloud-clear air interface)
- Fractal SLE curves in 2D and quasi-geostrophic turbulence
- Fractal interpolation method to reconstruct subrgid scales in LES – Scotti & Meneveau, (1999)

Fractal interpolation technique (FIT)

Stretching parameter d - the vertical stretching of the left and right segments of three interpolation points at each iteration.
The stretching parameter is related to the scaling exponent of the spectrum D as:

$$D = 1 + log_N \sum_{n=1}^{N} |d_n| \quad \approx \quad 5/3$$

where $N + 1$ is the number of anchor points [Orey (1970), Praskovsky et al. (1993), Scotti et al. (1995)]

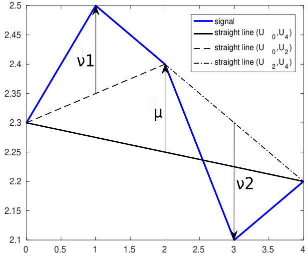


Figure: Estimation of the stretching parameters.

Stretching parameters

- $d = \pm 2^{-1/3}$ Scotti and Meneveau [Physica D, **127** 1999]
- $d = -0.887, -0.676$ Basu et al. [Phys. Rev. E, **70** 2004]
- spatially randomized $|d|$ with a prescribed Log-Poisson distribution Ding et al. Phys. Rev. E **82** 2010
- stretching parameters estimated directly from experimental data Akinlabi et al. [Flow, Turb. Comb **103** 2019]

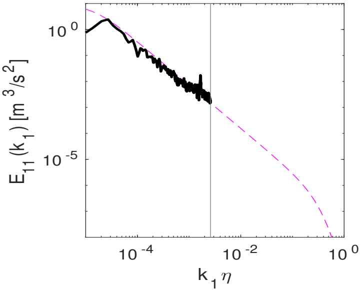


Figure: Velocity spectrum in stratocumulus cloud (POST campaign). black line: experiment, magenta line: theoretical form

Estimation of the stretching parameters

- Consecutive filtering of the velocity/ scalar field is performed
  Akinlabi et al. [Flow, Turb. Comb **103** 2019]
- Mazel & Hayes [IEEE Trans. on signal processing **40**, 1992] algorithm is used to calculate stretching coefficients at each step by comparison of the fields at resolution n and $n+1$

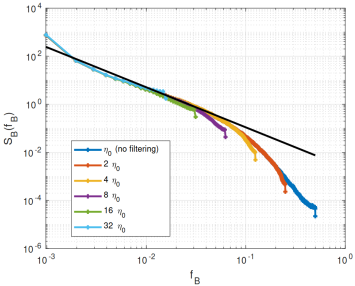


Figure: Frequency spectrum of buoyancy in stratocumulus cloud (based on DNS data of J. P. Mellado, *priv. comm*).

Estimation of the stretching parameters

- Stretching parameter *d* is a random variable and we determined its probability density function (PDF).
- For this we used various data from numerical and field experiments
- In the inertial range the PDF has a universal shape

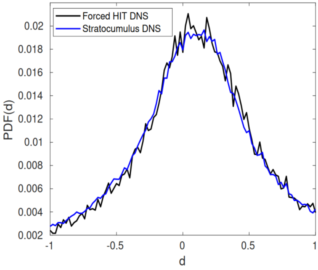


Figure: PDF of stretching parameters for velocity.

Estimation of the stretching parameters

- Universal PDF of *d* is also obtained for scalar fields - potential temperature and specific humidity
- it is consistent with the $-5/3$ scaling the inertial range,
- but it also accounts for the intermittency (scale-symmetry breaking)

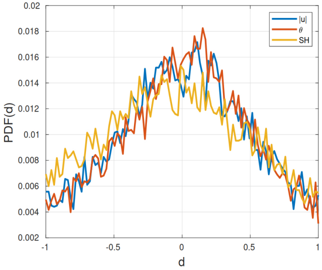


Figure: PDF of stretching parameters for velocity, potential temperature and specific humidity. Based on POST data (*E. Akinlabi, priv. comm.*).

Reconstruction of sub-grid velocity

Sub-grid velocity and scalar fields can be reconstructed with FIT

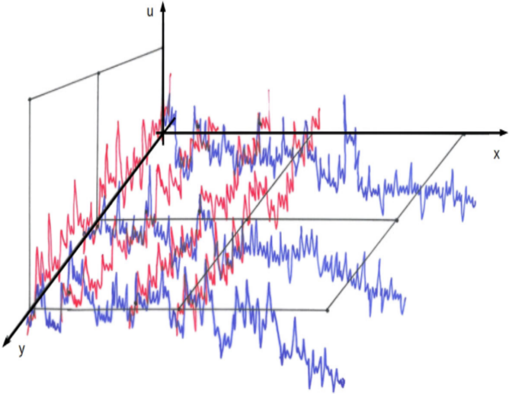


Figure: Reconstruction of a 2D field

Reconstruction of sub-grid velocity

Sub-grid velocity and scalar fields can be reconstructed with FIT

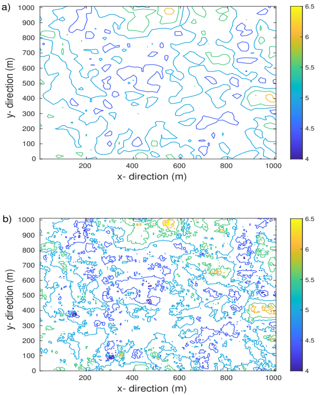


Figure: a) filtered LES velocity field, b) reconstructed field

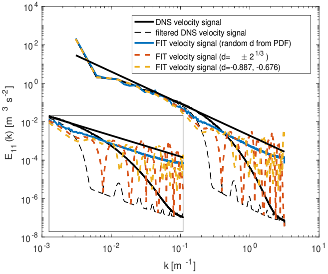


Figure: Reconstructed velocity spectra.

Reconstruction of sub-grid velocity

Intermittency in turbulence – non-Gaussian PDF of velocity increments

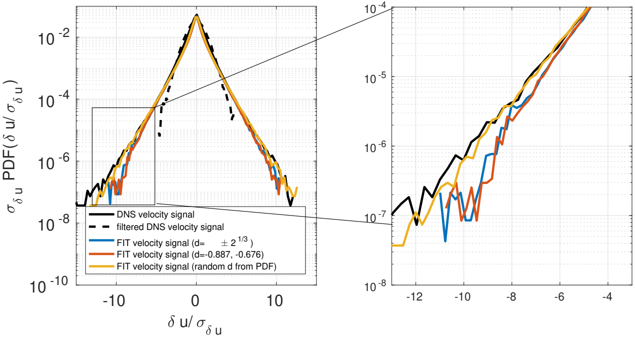


Figure: PDF of velocity increment $\delta u = u(x + r) - u(x)$. [Akinlabi et al., Flow, Turb. Comb., **103**, 2019]

Reconstruction of sub-grid velocity

Consider LES field with grid resolution Δ.
Test filtering with filter of the width 2Δ provides the 'residual' kinetic energy

$$k_r = \widehat{\tilde{u}_i \tilde{u}_i} - \widehat{\tilde{u}}_i \widehat{\tilde{u}}_i$$

which can be compared with results of fractal reconstruction back to resolution Δ.

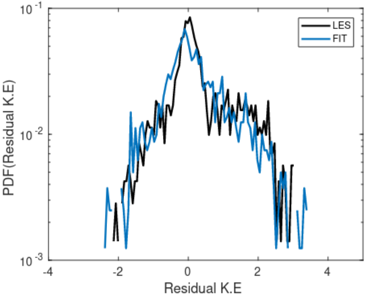


Figure: Subgrid kinetic energy for test-filtered LES compared with results of FIT reconstruction [E. Akinlabi PhD thesis, 2020]

Reconstruction of sub-grid velocity

Reconstructed FIT field is correlated in space but not in time.

Justification: A priori analysis of DNS data show short-range time correlation of the stretching parameters (of the order of τ_η)

comparable to the autocorrelation of velocity gradients.

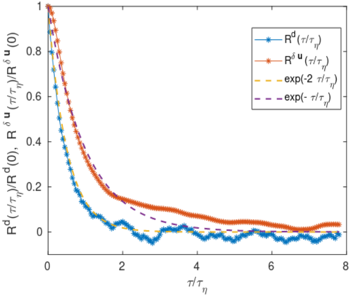


Figure: Correlation of stretching parameters and velocity gradients [E. Akinlabi PhD thesis, 2020]

Reconstruction of sub-grid velocity

In 1D single reconstruction step doubles number of grid points.

In 3D single reconstruction step increases number of grid points $2^3 = 8$ times

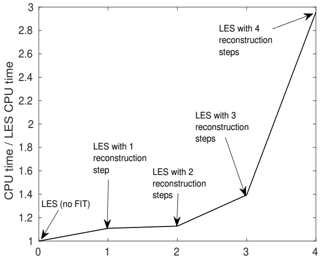


Figure: CPU time for 0,1,2,3 and 4 reconstruction steps.

Lagrangian particle tracking in the reconstructed velocity field

velocity field

$$\mathbf{U} = \mathbf{U}_{LES} + \mathbf{u}_{sgs}$$

scalar fields

$$\theta = \theta_{LES} + \theta_{sgs}$$

particle tracking

$$\frac{d\mathbf{X}}{dt} = \mathbf{U}_{LES} + \mathbf{u}_{sgs}$$

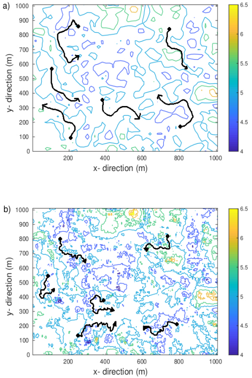


Figure: Illustrative picture of particle tracking in the filtered coarse-grained and in the reconstructed field.

Conclusions

- Fractal interpolation technique (FIT) with random stretching parameters used to reconstruct residual field in LES.
- FIT correctly predicts some rough features of sub-grid turbulence in the inertial range, like the intermittency and scaling.
- FIT can be coupled with Lagrangian models for particle tracking.