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Dynamics of the Thermal Vortex Ring:
Vortex–Dynamics Perspective

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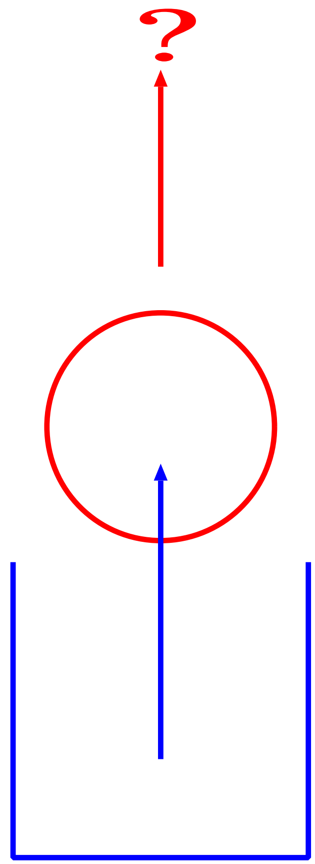
Outline:

1. **What is the Thermal Vortex Ring?**
2. **Vortex Dynamics**
3. **Simple Analytical Solution**
4. **General Formulation**
5. **Similarity Solutions**
6. **Conclusions**

What is the Thermal Vortex Ring?: How to Create It:



e.g., Release a buoyancy (thermal) anomaly at a bottom of an apparatus

What is the Thermal Vortex Ring?: Laboratory Example:

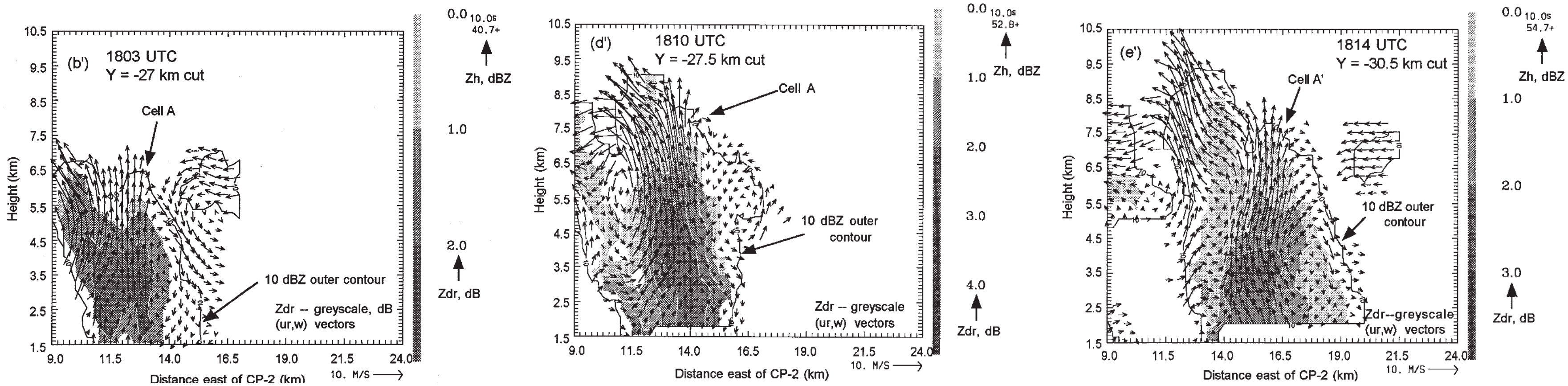
(Anna Gorska, Szymon Malinowski, Warsaw)

Thermal Vortex Ring: Relevance?:

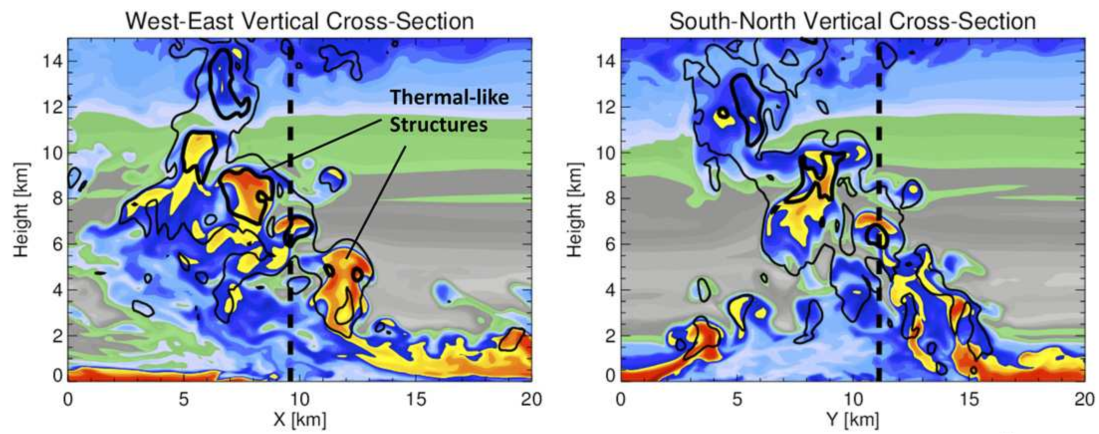
Basic Elements of Convection?

i.e.,

Convection Consists of Ensemble of Thermal Vortex Rings (Thermals)



Doppler Observation: (Bringi et al., 1991)



Numerical Simulation: (Morrison et al., 2020)

Basic Dynamics of the Thermal Vortex Ring:

Momentum Eq:

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p_d}{\partial z} - \frac{1}{\rho}\frac{\partial p_b}{\partial z} + b$$

Volume-Averaged Eq:

$$\frac{d\bar{w}}{dt} = -\hat{C}_d w^2 + \frac{b}{1+\gamma} + E$$

(cf., Levine 1959, Simpson and Wiggert 1969)

NB: Difficult to Develop a Closed Formulation

Alternative Approach: Vortex Dynamics:

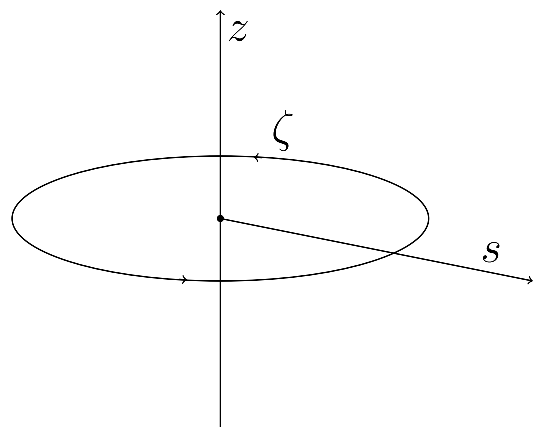
Momentum Eq:

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$$

Apply $\nabla\times$:

Vorticity Eq (azimuthal direction):

$$(\frac{\partial}{\partial t} + v_s\frac{\partial}{\partial s} + v_z\frac{\partial}{\partial z})\frac{\zeta}{s} = -\frac{1}{s}\frac{\partial b}{\partial s}$$



Vortex Dynamics:

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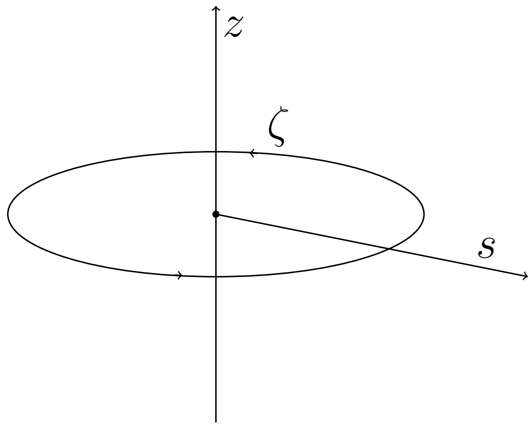

NB: ζ/s: c.f., potential vorticity
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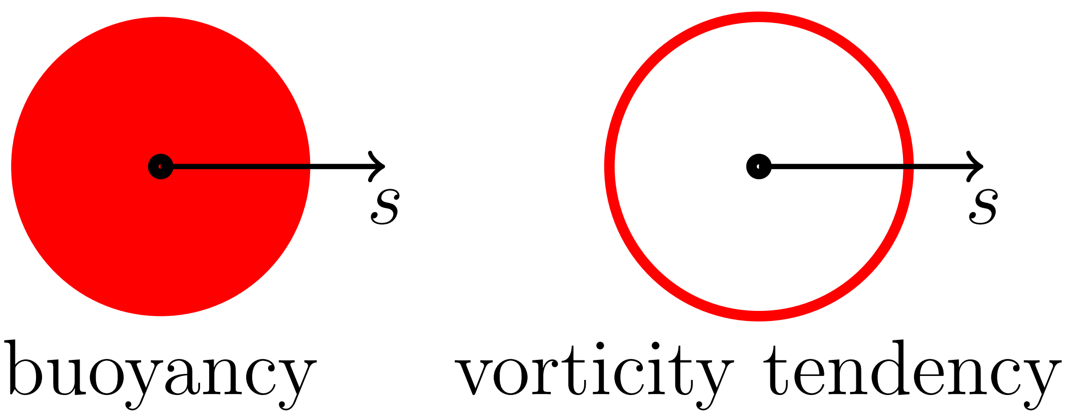


Closed System:

Vorticity Eq (azimuthal direction):

$$(\frac{\partial}{\partial t}+v_s\frac{\partial}{\partial s}+v_z\frac{\partial}{\partial z})\frac{\zeta}{s}=-\frac{1}{s}\frac{\partial b}{\partial s}$$

Buoyancy Eq:

$$(\frac{\partial}{\partial t}+v_s\frac{\partial}{\partial s}+v_z\frac{\partial}{\partial z})b=0$$

Seek: Steadily–Propagating Solution:

$$z=z'-ct,$$
$$v_z=v_z'-c.$$

Steadily–Propagating Solution:

$$z = z' - ct,$$
$$v_z = v_z' - c.$$

then

$$(v_s\frac{\partial}{\partial s} + v_z\frac{\partial}{\partial z})\frac{\zeta}{s} = -\frac{1}{s}\frac{\partial b}{\partial s}$$
$$(v_s\frac{\partial}{\partial s} + v_z\frac{\partial}{\partial z})b = 0$$

Stream Function:

$$v_z = -\frac{1}{s}\frac{\partial\psi}{\partial s}, v_s = \frac{1}{s}\frac{\partial\psi}{\partial z},$$
$$\zeta = \left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial^2}{\partial z^2}\right)\psi.$$

Steadily–Propagating Solution:

$$(v_s \frac{\partial}{\partial s} + v_z \frac{\partial}{\partial z}) \frac{\zeta}{s} = -\frac{1}{s} \frac{\partial b}{\partial s}$$

$$(v_s \frac{\partial}{\partial s} + v_z \frac{\partial}{\partial z}) b = 0$$

Stream Function:

$$v_z = -\frac{1}{s} \frac{\partial \psi}{\partial s}, v_s = \frac{1}{s} \frac{\partial \psi}{\partial z},$$

$$\zeta = \left(\frac{\partial}{\partial s} \frac{1}{s} \frac{\partial}{\partial s} + \frac{1}{s} \frac{\partial^2}{\partial z^2} \right) \psi.$$

then

$$J(\frac{\zeta}{s}, \psi) = -\frac{\partial b}{\partial s}, \quad J(b, \psi) = 0$$

where **Jacobian:**

$$J(a, b) = \frac{\partial a}{\partial s} \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \frac{\partial b}{\partial s}$$

Steadily–Propagating Solution:

$$J(\frac{\zeta}{s},\psi) = -\frac{\partial b}{\partial s}, \quad J(b,\psi) = 0$$

where **Jacobian:**

$$J(a,b) = \frac{\partial a}{\partial s}\frac{\partial b}{\partial z} - \frac{\partial a}{\partial z}\frac{\partial b}{\partial s}$$

then

$$b = \mathcal{F}(\psi), \quad \text{or} \;\; b = -\alpha\psi$$

&

$$J(Q,\psi) = 0, \quad Q = \frac{\zeta}{s} + \alpha z$$

(cf., QG Potential Vorticity on β-Plane)

Let:

$$Q = \begin{cases} Q_0 & r \leq R_b \\ 0 & r > R_b \end{cases}$$

(i.e., PV Patch)

Steadily–Propagating Solution:

$$J(Q, \psi) = 0, \quad Q = \frac{\zeta}{s} + \alpha z$$

(cf., QG Potential Vorticity on β-Plane)

Let:

$$Q = \begin{cases} Q_0 & r \leq R_b \\ 0 & r > R_b \end{cases}$$

or

$$\zeta = \begin{cases} Q_0\, s - \alpha s z & r \leq R_b \\ 0 & r > R_b \end{cases}$$

Steadily–Propagating Solution:

$$\zeta = \begin{cases} Q_0\, s - \alpha s z & r \le R_b \\ 0 & r > R_b \end{cases}$$

Let: $\psi = \bar{\psi} + \psi'$, **&**

$$\left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial^2}{\partial z^2}\right)\bar{\psi} = \begin{cases} Q_0\, s & r \le R_b \\ 0 & r > R_b \end{cases}$$

$$\left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial^2}{\partial z^2}\right)\psi' = \begin{cases} -\alpha s z & r \le R_b \\ 0 & r > R_b \end{cases}$$

NB:

$$\left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial^2}{\partial z^2}\right) s^l z^m r^n = [l(l-2)z^2 + m(m-1)s^2]s^{l-3}z^{m-2}r^n + n[2(l+m)+n-1]s^{l-1}z^m r^{n-2}$$

:A Closed Analytical Solution

($\bar{\psi}$: Hill's vortex), Except for $R_b = R_b(\theta)$.

$\bar{\psi}$: **Hill's Vortex**

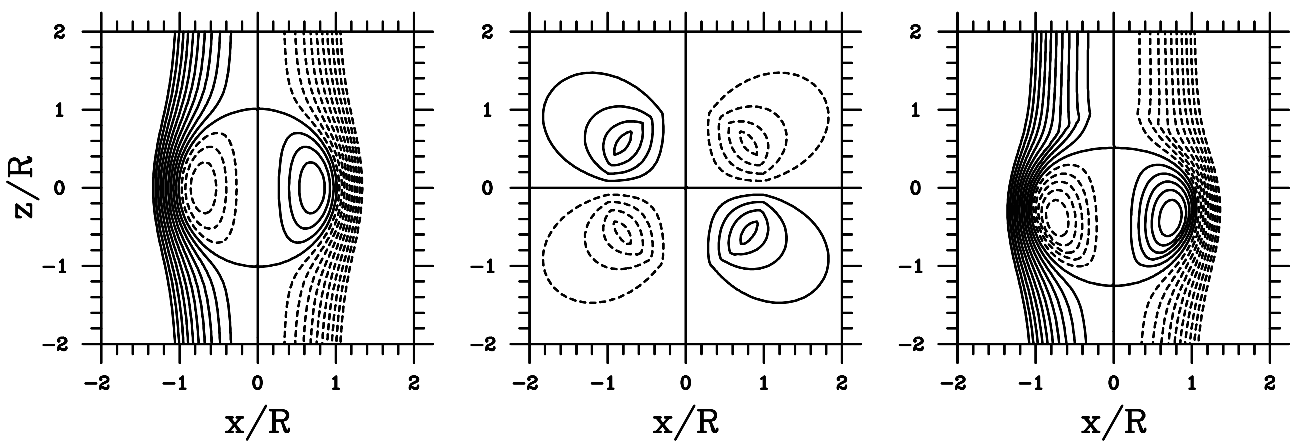
ψ': **Modification by Buoyancy**

Vortex–Ring Boundary:

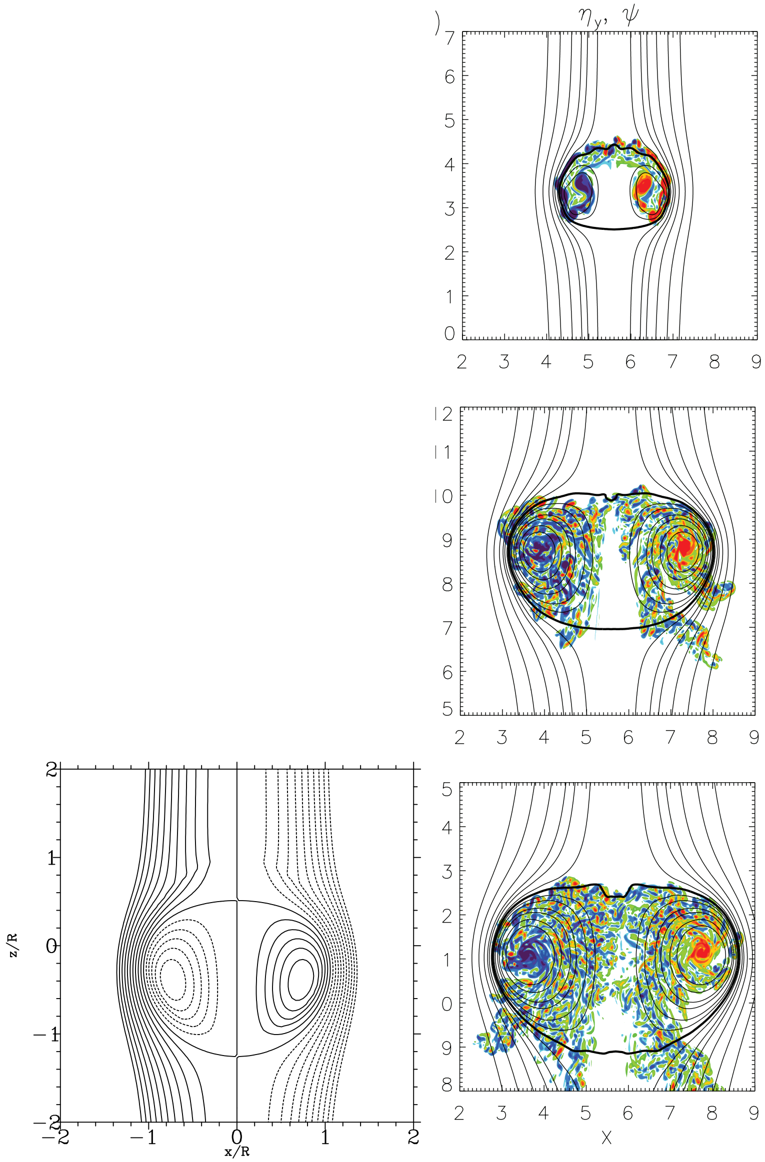
$$R_b/R \simeq 1 + \tilde{R}_1 \cos\theta + \tilde{R}_2 \cos^2\theta$$

where

$$\tilde{R}_1 = \frac{2\alpha R}{35A}, \qquad \tilde{R}_2 = -2\alpha^2 \tilde{R}_1^2, \qquad A < 0$$



Analytical and Numerical Resutls:



(Morrison,Jeevanje,Yano)

Interpretation:

Potential–Vorticity Conservation: $Q = \frac{\zeta}{s} + \alpha z$

i.e.,
Vorticity decreases with Height
due to Buoyancy

Similarity Solutions (Dimensional Analysis, Scorer 1957):

Basic Variables: $\bar{b}$**,** $w(=c)$**,** R**,** $\bar{z}$

Dimensional Consistency:

$$w = (f\bar{b}R)^{1/2}, \quad R = \mu\bar{z}$$

where f**: Froude number (constant)**

Conservation of Buoyancy Flux:

$$R^3\bar{b} = \text{const}$$

Assumption: Vortex Ring is Spherical

NB: $w = d\bar{z}/dt$**, or** $(d/dt)R^{1/2} \sim R$**:**

$$R \sim \bar{z} \sim t^{1/2}, \quad w \sim t^{-1/2}$$

Q: Deductive Derivation?

Vortex Dynamics:General Formulation:

$$(\frac{\partial}{\partial t} + v_s\frac{\partial}{\partial s} + v_z\frac{\partial}{\partial z})\frac{\zeta}{s} = -\frac{1}{s}\frac{\partial b}{\partial s}$$

Continuity:

$$\frac{1}{s}\frac{\partial}{\partial s}sv_s + \frac{\partial v_z}{\partial z} = 0$$

then

$$\frac{\partial \zeta}{\partial t} + \frac{\partial v_s\zeta}{\partial s} + \frac{\partial v_z\zeta}{\partial z} = -\frac{\partial b}{\partial s}.$$

Let: Solution = Intensity × Shape: i.e.,

$$\zeta(s,z) = \frac{\zeta_0}{R^2}\tilde{\zeta}(\xi,\eta), \quad b(s,z) = \bar{b}\tilde{b}(\xi,\eta),$$

$$v_s = \frac{\zeta_0}{R}\tilde{v}_s, \quad v_z = \frac{\zeta_0}{R}\tilde{v}_z$$

with $(\xi,\eta) = (s/R, z/R)$

cf., Green's function:

$$\left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial^2}{\partial z^2}\right) G(s,z|s_0,z_0) = \delta(s-s_0)\delta(z-z_0)$$

$$\psi = 2\pi \int_{-\infty}^{+\infty}\int_0^{+\infty} \zeta(s_0,z_0)G(s,z|s_0,z_0)s_0\, ds_0\, dz_0,$$

Propagation Speed:

$$w = \frac{\zeta_0}{R}\langle\tilde{v}_z\rangle_{r<R}$$

from Vortex Dynamics to Similarity Solutions:

Integral Quantities: $I_n = \langle s^n \zeta \rangle$

Impulse: $P = \langle s\zeta \rangle$, $n = 1$

$n = -1$:

$$\frac{d}{dt}(Z\zeta_0) = 0$$

if the shape factor, $\dot{Z} = 0$**:** $\dot{\zeta}_0 = 0$ **[original]**

$n = 1$**:** $\dot{P} = V\bar{b} \equiv F$ **&** $P = \gamma R^2 \zeta_0$**:**

$$\gamma R^2 \dot{\zeta}_0 + \gamma \zeta_0 \frac{dR^2}{dt} = F$$

if $\dot{Z} = 0$**:** $\dot{\zeta}_0 = 0$**:**

$$\gamma \zeta_0 \frac{dR^2}{dt} = F, \qquad R^2 \sim t \qquad \text{(cf., Turner1957)}$$

or $R \sim t^{1/2}$**, etc:** **Similarity Solutions!**

Conclusions:

- **Vortex equation facilitates understanding of dynamics of thermal vortex ring, more easily than the momentum equation**
- **Simple analytical solution: basic features of numerically–simulated thermal vortex rings**
- **General formulation for vortex dynamics of thermal vortex rings**
- **Deductive derivation of the similarity solution**
- **Diagnostic framework of numerical simulations and experiments**