Rigorous derivation of magneto-Boussinesq approximation with non-local term

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Joint work with P. Gwiazda, A. Wróblewska-Kamińska

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The model

$D = \mathbb{T}^2 \times (0,1)$ periodic strip. For $T > 0$ in $(0,T) \times D$,

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\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla p(\varrho, \vartheta) &= \frac{1}{\mathrm{Fr}^2} \varrho \nabla G + \frac{1}{\mathrm{Al}^2} \operatorname{curl} \mathbf{B} \times \mathbf{B}, \\
\partial_t \mathbf{B} + \operatorname{curl}(\mathbf{B} \times \mathbf{u}) + \operatorname{curl}(\zeta(\vartheta) \operatorname{curl} \mathbf{B}) &= 0, \\
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\partial_t(\varrho s) + \operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div} \frac{\mathbf{q}}{\vartheta} &= \sigma, \\
\frac{1}{\vartheta} \left(\mathrm{Ma}^2 \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} + \frac{\mathrm{Ma}^2}{\mathrm{Al}^2} \zeta(\vartheta) |\operatorname{curl} \mathbf{B}|^2 \right) &= \sigma
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- $\mathrm{Ma} = u_c / \sqrt{p_c/\varrho_c} = \varepsilon$, $\mathrm{Fr} = u_c / \sqrt{g L_c} = \varepsilon^{\frac{1}{2}}$, $\mathrm{Al} = u_c / (B_c / \sqrt{\varrho_c}) = \varepsilon$

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Boundary conditions on ∂D:

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad [\mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{n}] \times \mathbf{n} = 0, \qquad \mathbf{B} \times \mathbf{n} = 0, \qquad \vartheta = \overline{\vartheta} + \varepsilon \vartheta_B$$

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Question: What happens when $\varepsilon \to 0$?

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- in turn, can write $\varrho_\varepsilon = \overline{\varrho} + \varepsilon\varrho^1,\; \vartheta_\varepsilon = \overline{\vartheta} + \varepsilon\vartheta^1,\; \mathbf{B}_\varepsilon = \overline{\mathbf{B}} + \varepsilon\mathbf{B}^1$

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- $\mathbf{B}_\varepsilon \times \mathbf{n} = 0$ and $\mathbf{n} = \pm\mathbf{e}_3 \;\Rightarrow \overline{\mathbf{B}} = (0, 0, \overline{b})$, and $\mathbf{B}^1 = (0, 0, b^1)$; by div $\mathbf{B}^1 = 0$, we have $b^1 = b^1(t, x_1, x_2)$

Formal derivation II: CE, form of $\mathbf{U}$, IE

- CE:

$$0 = \partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) \to \partial_t \overline{\varrho} + \operatorname{div}(\overline{\varrho}\mathbf{U}) \Rightarrow \boxed{\operatorname{div} \mathbf{U} = 0}$$

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- IE: Plug in $\mathbf{B}_\varepsilon = \overline{\mathbf{B}} + \varepsilon \mathbf{B}^1$ to get

$$\varepsilon \partial_t \mathbf{B}^1 + \operatorname{curl}((\overline{\mathbf{B}} + \varepsilon \mathbf{B}^1) \times \mathbf{u}_\varepsilon) + \varepsilon \operatorname{curl}(\zeta(\overline{\vartheta} + \varepsilon \vartheta^1) \operatorname{curl} \mathbf{B}^1) = 0$$

$$\Rightarrow \; \operatorname{curl}(\overline{\mathbf{B}} \times \mathbf{U}) = (\mathbf{U}\cdot\nabla)\overline{\mathbf{B}} - (\overline{\mathbf{B}}\cdot\nabla)\mathbf{U} \overset{!}{=} 0 \;\Rightarrow\; \boxed{\partial_3 \mathbf{U} = 0}$$

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- BC on $\mathbf{u}_\varepsilon$: $0 = \mathbf{u}_\varepsilon \cdot \mathbf{n} \to \mathbf{U} \cdot \mathbf{n} \;\Rightarrow\; \boxed{\mathbf{U} = (U_1, U_2, 0)(t, x_1, x_2)}$

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- Know: $\int_D \varrho^1 \, \mathrm{d}x = 0, \ \int_D \mathbf{B}^1 \, \mathrm{d}x = 0$; assume: $\int_D G \, \mathrm{d}x = 0$, then $\chi(t) = \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \fint_D \vartheta^1 \, \mathrm{d}x$ and Boussinesq reads

$$\boxed{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \varrho^1 + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \vartheta^1 + \overline{\mathbf{B}} \cdot \mathbf{B}^1 = \overline{\varrho} G + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \fint_D \vartheta^1 \, \mathrm{d}x.}$$

Formal derivation IV: ME, part 1

- ME again: recall

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \operatorname{div} \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon)$$
$$= -\frac{1}{\varepsilon^2} \nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{\varepsilon} \varrho_\varepsilon \nabla G + \frac{1}{\varepsilon^2} \operatorname{curl} \mathbf{B}_\varepsilon \times \mathbf{B}_\epsilon$$

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- LHS: as $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon) \to (\overline{\varrho}, \overline{\vartheta}, \mathbf{U})$,

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$$\to \overline{\varrho}(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) - \operatorname{div} \mathbb{S}(\overline{\vartheta}, \nabla \mathbf{U})$$
$$= \overline{\varrho}(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) - \mu(\overline{\vartheta})\nabla^2 \mathbf{U}$$

Formal derivation V: ME, part 2

- RHS: as before,

$$\operatorname{curl}\mathbf{B}_\varepsilon \times \mathbf{B}_\varepsilon = \operatorname{curl}(\overline{\mathbf{B}} + \varepsilon\mathbf{B}^1) \times (\overline{\mathbf{B}} + \varepsilon\mathbf{B}^1)$$
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- Hence,

$$-\frac{1}{\varepsilon^2} \nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{\varepsilon} \varrho_\varepsilon \nabla G + \frac{1}{\varepsilon^2} \operatorname{curl}\mathbf{B}_\varepsilon \times \mathbf{B}_\epsilon$$
$$= -\frac{1}{\varepsilon} \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \nabla \varrho^1 + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \nabla \vartheta^1 \right) + \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \nabla G + \frac{1}{\varepsilon} \overline{\varrho} \nabla G$$
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Formal derivation V: ME, part 2

- RHS: as before,

$$\operatorname{curl}\mathbf{B}_\varepsilon \times \mathbf{B}_\varepsilon = \operatorname{curl}(\overline{\mathbf{B}} + \varepsilon\mathbf{B}^1) \times (\overline{\mathbf{B}} + \varepsilon\mathbf{B}^1)$$
$$= \varepsilon \operatorname{curl}\mathbf{B}^1 \times \overline{\mathbf{B}} + \varepsilon^2 \operatorname{curl}\mathbf{B}^1 \times \mathbf{B}^1 = -\varepsilon\nabla(\overline{\mathbf{B}}\cdot\mathbf{B}^1) - \varepsilon^2\nabla\frac{1}{2}|\mathbf{B}^1|^2$$

- Moreover,

$$\nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) = \varepsilon\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})\nabla\varrho^1 + \varepsilon\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\nabla\vartheta^1$$

- Hence,

$$-\frac{1}{\varepsilon^2}\nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{\varepsilon}\varrho_\varepsilon\nabla G + \frac{1}{\varepsilon^2}\operatorname{curl}\mathbf{B}_\varepsilon \times \mathbf{B}_\epsilon$$
$$= -\frac{1}{\varepsilon}\left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})\nabla\varrho^1 + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\nabla\vartheta^1\right) + \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\nabla G + \frac{1}{\varepsilon}\overline{\varrho}\nabla G$$
$$-\frac{1}{\varepsilon}\nabla(\overline{\mathbf{B}}\cdot\mathbf{B}^1) - \nabla\frac{1}{2}|\mathbf{B}^1|^2$$

- By Boussinesq relation,

$$-\frac{1}{\varepsilon^2}\nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{\varepsilon}\varrho_\varepsilon\nabla G + \frac{1}{\varepsilon^2}\operatorname{curl}\mathbf{B}_\varepsilon \times \mathbf{B}_\epsilon$$
$$\to \varrho^1\nabla G - \nabla\frac{1}{2}|\mathbf{B}^1|^2 - \nabla\pi = \varrho^1\nabla G - \nabla\Pi$$

Formal derivation VI: HE, part 1

- HE: recall

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}\, \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla\vartheta_\varepsilon)}{\vartheta_\varepsilon}$$
$$= \frac{1}{\vartheta_\varepsilon}\left(\varepsilon^2 \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\mathbf{u}_\varepsilon - \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla\vartheta_\varepsilon)\cdot\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon} + \zeta(\vartheta_\varepsilon)|\,\mathrm{curl}\,\mathbf{B}_\varepsilon|^2\right)$$

Formal derivation VI: HE, part 1

- HE: recall

$$\varrho_\varepsilon \big(\partial_t s(\varrho_\varepsilon, \vartheta_\varepsilon)\big) + \operatorname{div}\big(s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon\big) - \operatorname{div} \frac{\kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon} = \mathcal{O}(\varepsilon^2)$$

Formal derivation VI: HE, part 1

- HE: recall

$$\varrho_\varepsilon\big(\partial_t s(\varrho_\varepsilon,\vartheta_\varepsilon)\big) + \operatorname{div}(s(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon)\big) - \operatorname{div}\frac{\kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon} = \mathcal{O}(\varepsilon^2)$$

- Expanding (recall $\varrho_\varepsilon = \overline{\varrho} + \varepsilon\varrho^1$, $\vartheta_\varepsilon = \overline{\vartheta} + \varepsilon\vartheta^1$)

$$s(\varrho_\varepsilon,\vartheta_\varepsilon) = s(\overline{\varrho},\overline{\vartheta}) + \varepsilon\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^1 + \varepsilon\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^1 + \mathcal{O}(\varepsilon^2),$$

we get ($\partial_*\bar{s} = \partial_* s(\overline{\varrho},\overline{\vartheta})$)

$$(\overline{\varrho}+\varepsilon\varrho^1)\big(\varepsilon\partial_\varrho\bar{s}\partial_t\varrho^1 + \varepsilon\partial_\vartheta\bar{s}\partial_t\vartheta^1 + \operatorname{div}\big[\mathbf{u}_\varepsilon\big(\varepsilon\partial_\varrho\bar{s}\varrho^1 + \varepsilon\partial_\vartheta\bar{s}\vartheta^1\big)\big]\big) - \varepsilon\operatorname{div}\frac{\kappa(\vartheta_\varepsilon)\nabla\vartheta^1}{\vartheta_\varepsilon} = \mathcal{O}(\varepsilon^2),$$

Formal derivation VI: HE, part 1

- HE: recall

$$\varrho_\varepsilon\big(\partial_t s(\varrho_\varepsilon,\vartheta_\varepsilon)\big)+\operatorname{div}(s(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon)\big)-\operatorname{div}\frac{\kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon}=\mathcal{O}(\varepsilon^2)$$

- Expanding (recall $\varrho_\varepsilon=\overline{\varrho}+\varepsilon\varrho^1$, $\vartheta_\varepsilon=\overline{\vartheta}+\varepsilon\vartheta^1$)

$$s(\varrho_\varepsilon,\vartheta_\varepsilon)=s(\overline{\varrho},\overline{\vartheta})+\varepsilon\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^1+\varepsilon\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^1+\mathcal{O}(\varepsilon^2),$$

we get ($\partial_*\bar{s}=\partial_* s(\overline{\varrho},\overline{\vartheta})$)

$$(\overline{\varrho}+\varepsilon\varrho^1)\big(\varepsilon\partial_\varrho\bar{s}\partial_t\varrho^1+\varepsilon\partial_\vartheta\bar{s}\partial_t\vartheta^1+\operatorname{div}\left[\mathbf{u}_\varepsilon\big(\varepsilon\partial_\varrho\bar{s}\varrho^1+\varepsilon\partial_\vartheta\bar{s}\vartheta^1\big)\right]\big)$$
$$-\varepsilon\operatorname{div}\frac{\kappa(\vartheta_\varepsilon)\nabla\vartheta^1}{\vartheta_\varepsilon}=\mathcal{O}(\varepsilon^2),$$

in turn for $\varepsilon\to 0$

$$\boxed{\overline{\varrho}\partial_t\big(\partial_\varrho\bar{s}\varrho^1+\partial_\vartheta\bar{s}\vartheta^1\big)+\overline{\varrho}\operatorname{div}\left[\mathbf{U}\big(\partial_\varrho\bar{s}\varrho^1+\partial_\vartheta\bar{s}\vartheta^1\big)\right]-\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla^2\vartheta^1=0.}$$

Formal derivation VII: HE, part 2

- Gibbs' relation $\vartheta Ds = De + pD(1/\varrho)$ yields

$$\vartheta \partial_\vartheta s = \partial_\vartheta e, \qquad \vartheta \partial_\varrho s = \partial_\varrho e - \frac{p}{\varrho^2}.$$

Formal derivation VII: HE, part 2

- Gibbs' relation $\vartheta Ds = De + pD(1/\varrho)$ yields

$$\vartheta \partial_\vartheta s = \partial_\vartheta e, \qquad \vartheta \partial_\varrho s = \partial_\varrho e - \frac{p}{\varrho^2}.$$

- Taking cross-derivatives wrt. ϱ and ϑ, we get

$$\vartheta \partial^2_{\varrho\vartheta} s = \partial^2_{\varrho\vartheta} e, \quad \partial_\varrho s + \vartheta \partial^2_{\varrho\vartheta} s = \partial^2_{\varrho\vartheta} e - \frac{\partial_\vartheta p}{\varrho^2} \;\Rightarrow\; \boxed{\vartheta \partial_\varrho s = -\frac{\vartheta}{\varrho^2}\partial_\vartheta p}$$

Formal derivation VII: HE, part 2

- Gibbs' relation $\vartheta Ds = De + pD(1/\varrho)$ yields

$$\vartheta \partial_\vartheta s = \partial_\vartheta e, \qquad \vartheta \partial_\varrho s = \partial_\varrho e - \frac{p}{\varrho^2}.$$

- Taking cross-derivatives wrt. ϱ and ϑ, we get

$$\vartheta \partial^2_{\varrho\vartheta} s = \partial^2_{\varrho\vartheta} e, \quad \partial_\varrho s + \vartheta \partial^2_{\varrho\vartheta} s = \partial^2_{\varrho\vartheta} e - \frac{\partial_\vartheta p}{\varrho^2} \quad \Rightarrow \boxed{\vartheta \partial_\varrho s = -\frac{\vartheta}{\varrho^2} \partial_\vartheta p}$$

- from BR, we have

$$\varrho^1 = \frac{\overline{\varrho}}{\partial_\varrho p} G - \frac{1}{\partial_\varrho p} \overline{\mathbf{B}} \cdot \mathbf{B}^1 - \frac{\partial_\vartheta p}{\partial_\varrho p} \left(\vartheta^1 - \fint_D \vartheta^1 \, \mathrm{d}x \right)$$

Formal derivation VIII: HE, part 3

- Collecting equations:

$$\overline{\varrho}\partial_t\left(\partial_\varrho\overline{s}\varrho^1 + \partial_\vartheta\overline{s}\vartheta^1\right) + \overline{\varrho}\,\mathrm{div}\left[\mathbf{U}\left(\partial_\varrho\overline{s}\varrho^1 + \partial_\vartheta\overline{s}\vartheta^1\right)\right] - \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla^2\vartheta^1 = 0$$

$$\overline{\vartheta}\partial_\vartheta\overline{s} = \partial_\vartheta e(\overline{\varrho},\overline{\vartheta}), \quad \overline{\vartheta}\partial_\varrho\overline{s} = -\frac{\overline{\vartheta}}{\overline{\varrho}^2}\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$$

$$\varrho^1 = \frac{\overline{\varrho}}{\partial_\varrho p}G - \frac{1}{\partial_\varrho p}\overline{\mathbf{B}}\cdot\mathbf{B}^1 - \frac{\partial_\vartheta p}{\partial_\varrho p}\left(\vartheta^1 - \fint_D \vartheta^1\,\mathrm{d}x\right)$$

Formal derivation VIII: HE, part 3

- Collecting equations:

$$\overline{\varrho}\partial_t\big(\partial_\varrho \overline{s}\varrho^1 + \partial_\vartheta \overline{s}\vartheta^1\big) + \overline{\varrho}\,\mathrm{div}\left[\mathbf{U}\big(\partial_\varrho \overline{s}\varrho^1 + \partial_\vartheta \overline{s}\vartheta^1\big)\right] - \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\nabla^2\vartheta^1 = 0$$

$$\overline{\vartheta}\partial_\vartheta \overline{s} = \partial_\vartheta e(\overline{\varrho},\overline{\vartheta}), \quad \overline{\vartheta}\partial_\varrho \overline{s} = -\frac{\overline{\vartheta}}{\overline{\varrho}^2}\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$$

$$\varrho^1 = \frac{\overline{\varrho}}{\partial_\varrho p}G - \frac{1}{\partial_\varrho p}\overline{\mathbf{B}}\cdot\mathbf{B}^1 - \frac{\partial_\vartheta p}{\partial_\varrho p}\left(\vartheta^1 - \fint_D \vartheta^1\,\mathrm{d}x\right)$$

- Putting all together, we find

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})(\partial_t\vartheta^1 + \mathbf{U}\cdot\nabla\vartheta^1) - \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\mathbf{U}\cdot\nabla G - \kappa(\overline{\vartheta})\nabla^2\vartheta^1$$
$$= \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\partial_t\fint_D \vartheta^1\,\mathrm{d}x - \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\big(\partial_t(\overline{\mathbf{B}}\cdot\mathbf{B}^1) + \mathbf{U}\cdot\nabla(\overline{\mathbf{B}}\cdot\mathbf{B}^1)\big),$$

where

$$\alpha(\overline{\varrho},\overline{\vartheta}) = \frac{1}{\overline{\varrho}}\frac{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}, \quad c_p(\overline{\varrho},\overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho},\overline{\vartheta}) + \frac{\overline{\vartheta}}{\overline{\varrho}}\alpha(\overline{\varrho},\overline{\vartheta})\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})$$

Target system

$$\operatorname{div} \mathbf{U} = 0, \quad \operatorname{div} \mathbf{B}^1 = 0,$$

$$\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})\varrho^1 + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\vartheta^1 + \overline{\mathbf{B}} \cdot \mathbf{B}^1 = \overline{\varrho} G + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \fint_D \vartheta^1 \, \mathrm{d}x$$

$$\overline{\varrho}(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) - \mu(\overline{\vartheta})\nabla^2 \mathbf{U} + \nabla \Pi = \varrho^1 \nabla G,$$

$$\partial_t \mathbf{B}^1 + \operatorname{curl}(\mathbf{B}^1 \times \mathbf{U}) + \operatorname{curl}(\zeta(\overline{\vartheta}) \operatorname{curl} \mathbf{B}^1) = 0,$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})(\partial_t \vartheta^1 + \mathbf{U} \cdot \nabla \vartheta^1) - \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho}, \overline{\vartheta})\mathbf{U} \cdot \nabla G - \kappa(\overline{\vartheta})\nabla^2 \vartheta^1$$

$$= \overline{\vartheta}\alpha(\overline{\varrho}, \overline{\vartheta})\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\partial_t \fint_D \vartheta^1 \, \mathrm{d}x - \overline{\vartheta}\alpha(\overline{\varrho}, \overline{\vartheta})\big(\partial_t(\overline{\mathbf{B}} \cdot \mathbf{B}^1) + \mathbf{U} \cdot \nabla(\overline{\mathbf{B}} \cdot \mathbf{B}^1)\big),$$

where

$$\alpha(\overline{\varrho}, \overline{\vartheta}) = \frac{1}{\overline{\varrho}} \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})}, \quad c_p(\overline{\varrho}, \overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho}, \overline{\vartheta}) + \frac{\overline{\vartheta}}{\overline{\varrho}}\alpha(\overline{\varrho}, \overline{\vartheta})\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})$$

BC: $\vartheta^1|_{\partial D} = \vartheta_B$ (recall $\vartheta_\varepsilon|_{\partial D} = \overline{\vartheta} + \varepsilon \vartheta_B$)

Target system

Some modifications for magnetic field:

$$
\begin{aligned}
0 &= \partial_t \mathbf{B}^1 + \operatorname{curl}(\mathbf{B}^1 \times \mathbf{U}) + \operatorname{curl}(\zeta(\overline{\vartheta}) \operatorname{curl} \mathbf{B}^1) \\
&= \partial_t \mathbf{B}^1 + (\mathbf{U} \cdot \nabla)\mathbf{B}^1 - (\mathbf{B}^1 \cdot \nabla)\mathbf{U} - \zeta(\overline{\vartheta})\nabla^2 \mathbf{B}^1
\end{aligned}
$$

Target system

Some modifications for magnetic field:

$$
\begin{aligned}
0 &= \partial_t \mathbf{B}^1 + \operatorname{curl}(\mathbf{B}^1 \times \mathbf{U}) + \operatorname{curl}(\zeta(\overline{\vartheta}) \operatorname{curl} \mathbf{B}^1) \\
&= \partial_t \mathbf{B}^1 + (\mathbf{U} \cdot \nabla)\mathbf{B}^1 - (\mathbf{B}^1 \cdot \nabla)\mathbf{U} - \zeta(\overline{\vartheta})\nabla^2 \mathbf{B}^1
\end{aligned}
$$

Hence, also

$$
\begin{aligned}
\partial_t(\overline{\mathbf{B}} \cdot \mathbf{B}^1) + \mathbf{U} \cdot \nabla(\overline{\mathbf{B}} \cdot \mathbf{B}^1) &= \overline{\mathbf{B}} \cdot (\partial_t \mathbf{B}^1 + (\mathbf{U} \cdot \nabla)\mathbf{B}^1) \\
&= \overline{\mathbf{B}} \cdot ((\mathbf{B}^1 \cdot \nabla)\mathbf{U} + \zeta(\overline{\vartheta})\nabla^2 \mathbf{B}^1) \\
&= \underbrace{(\mathbf{B}^1 \cdot \nabla)(\overline{\mathbf{B}} \cdot \mathbf{U})}_{=0 \text{ by } \overline{\mathbf{B}} \perp \mathbf{U}} + \zeta(\overline{\vartheta})\nabla^2(\overline{\mathbf{B}} \cdot \mathbf{B}^1)
\end{aligned}
$$

Target system

Some modifications for magnetic field:

$$\begin{aligned} 0 &= \partial_t \mathbf{B}^1 + \operatorname{curl}(\mathbf{B}^1 \times \mathbf{U}) + \operatorname{curl}(\zeta(\overline{\vartheta}) \operatorname{curl} \mathbf{B}^1) \\ &= \partial_t \mathbf{B}^1 + (\mathbf{U} \cdot \nabla)\mathbf{B}^1 - (\mathbf{B}^1 \cdot \nabla)\mathbf{U} - \zeta(\overline{\vartheta})\nabla^2 \mathbf{B}^1 \end{aligned}$$

Hence, also

$$\begin{aligned} \partial_t(\overline{\mathbf{B}} \cdot \mathbf{B}^1) + \mathbf{U} \cdot \nabla(\overline{\mathbf{B}} \cdot \mathbf{B}^1) &= \overline{\mathbf{B}} \cdot (\partial_t \mathbf{B}^1 + (\mathbf{U} \cdot \nabla)\mathbf{B}^1) \\ &= \overline{\mathbf{B}} \cdot ((\mathbf{B}^1 \cdot \nabla)\mathbf{U} + \zeta(\overline{\vartheta})\nabla^2 \mathbf{B}^1) \\ &= \underbrace{(\mathbf{B}^1 \cdot \nabla)(\overline{\mathbf{B}} \cdot \mathbf{U})}_{=0 \text{ by } \overline{\mathbf{B}} \perp \mathbf{U}} + \zeta(\overline{\vartheta})\nabla^2(\overline{\mathbf{B}} \cdot \mathbf{B}^1) \end{aligned}$$

Final HE:

$$\begin{aligned} \overline{\varrho} c_p(\partial_t \vartheta^1 + \mathbf{U} \cdot \nabla \vartheta^1) - \overline{\varrho}\overline{\vartheta}\alpha \mathbf{U} \cdot \nabla G + \overline{\vartheta}\alpha\zeta\nabla^2(\overline{\mathbf{B}} \cdot \mathbf{B}^1) - \kappa\nabla^2\vartheta^1 \\ = \overline{\vartheta}\alpha\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\partial_t \fint_D \vartheta^1 \, \mathrm{d}x \end{aligned}$$

Relative energy

$$E\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\;\middle|\;r,\Theta,\mathbf{U},\mathbf{H}\right)=\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{\varepsilon^2}\frac{1}{2}|\mathbf{B}-\mathbf{H}|^2$$
$$+\frac{1}{\varepsilon^2}\Big[\varrho e(\varrho,\vartheta)-\Theta\Big(\varrho s(\varrho,\vartheta)-rs(r,\Theta)\Big)$$
$$-\left(e(r,\Theta)-\Theta s(r,\Theta)+\frac{p(r,\Theta)}{r}\right)(\varrho-r)-re(r,\Theta)\Big]$$

Relative energy

Relative energy inequality:

$$
\begin{aligned}
&\left[\int_D E\left(\varrho, \vartheta, \mathbf{u}, \mathbf{B} \,\middle|\, r, \Theta, \mathbf{U}, \mathbf{H}\right) \,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&+\int_0^\tau \int_D \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - \frac{1}{\varepsilon^2}\frac{\mathbf{q}(\vartheta, \nabla\vartheta)\cdot\nabla\vartheta}{\vartheta} + \frac{1}{\varepsilon^2}\zeta(\vartheta)|\operatorname{curl}\mathbf{B}|^2\right) \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant -\int_0^\tau \int_D \left(\varrho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U}) + \frac{1}{\varepsilon^2}p(\varrho, \vartheta)\mathbb{I} - \mathbb{S}(\vartheta, \nabla\mathbf{u})\right) : \nabla\mathbf{U} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon^2}\int_0^\tau \int_D (\operatorname{curl}\mathbf{B}\times\mathbf{B})\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau \int_D \varrho\left(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U} - \frac{1}{\varepsilon}\nabla G\right)\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon^2}\int_0^\tau \int_D \left(\varrho\Big(s(\varrho, \vartheta) - s(r, \Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho, \vartheta) - s(r, \Theta)\Big)\mathbf{u}\cdot\nabla\Theta + \frac{\mathbf{q}(\vartheta, \nabla\vartheta)}{\vartheta}\cdot\nabla\Theta\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon^2}\int_0^\tau \int_D \left(\left(1 - \frac{\varrho}{r}\right)\partial_t p(r, \Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla p(r, \Theta)\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon^2}\int_0^\tau \int_D \Big(\mathbf{B}\cdot\partial_t\mathbf{H} - (\mathbf{B}\times\mathbf{u})\cdot\operatorname{curl}\mathbf{H} - \zeta(\vartheta)\operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{H}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon^2}\int_0^\tau \int_D \mathbf{H}\cdot\partial_t\mathbf{H}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

Relative energy

Relative energy inequality:

$$\left[\int_D E\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\;\middle|\; r,\Theta,\mathbf{U},\mathbf{H}\right)\;\mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.}$$

$$\leqslant \int_0^\tau \int_D \text{sth}\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\;\middle|\; r,\Theta,\mathbf{U},\mathbf{H}\right)\;\mathrm{d}x\;\mathrm{d}t$$

Relative energy

Relative energy inequality:

$$\left[\int_D E\left(\varrho, \vartheta, \mathbf{u}, \mathbf{B} \,\middle|\, r, \Theta, \mathbf{U}, \mathbf{H}\right)\, \mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.}$$
$$\leqslant \int_0^\tau \int_D \text{sth}\left(\varrho, \vartheta, \mathbf{u}, \mathbf{B} \,\middle|\, r, \Theta, \mathbf{U}, \mathbf{H}\right)\, \mathrm{d}x\, \mathrm{d}t$$

Goal: Grönwall argument, once getting

$$\left[\int_D E\left(\varrho, \vartheta, \mathbf{u}, \mathbf{B} \,\middle|\, r, \Theta, \mathbf{U}, \mathbf{H}\right)\, \mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.}$$
$$\leqslant C \int_0^\tau \int_D E\left(\varrho, \vartheta, \mathbf{u}, \mathbf{B} \,\middle|\, r, \Theta, \mathbf{U}, \mathbf{H}\right)\, \mathrm{d}x\, \mathrm{d}t + \text{small error}$$

Relative energy

Relative energy inequality:

$$\left[\int_D E\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\ \middle|\ r,\Theta,\mathbf{U},\mathbf{H}\right)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.}$$
$$\leqslant \int_0^\tau \int_D \text{sth}\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\ \middle|\ r,\Theta,\mathbf{U},\mathbf{H}\right)\,\mathrm{d}x\,\mathrm{d}t$$

Goal: Grönwall argument, once getting

$$\left[\int_D E\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\ \middle|\ r,\Theta,\mathbf{U},\mathbf{H}\right)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.}$$
$$\leqslant C\int_0^\tau \int_D E\left(\varrho,\vartheta,\mathbf{u},\mathbf{B}\ \middle|\ r,\Theta,\mathbf{U},\mathbf{H}\right)\,\mathrm{d}x\,\mathrm{d}t + \text{small error}$$

Idea: Consider

$$E_\varepsilon = E\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{B}_\varepsilon\ \middle|\ \overline{\varrho}+\varepsilon\varrho^1,\overline{\vartheta}+\varepsilon\vartheta^1,\mathbf{U},\overline{\mathbf{B}}+\varepsilon\mathbf{B}^1\right)$$

Convergence

Outcome:

$$\left[\int_D E_\varepsilon \,\mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.} \leqslant C\int_0^\tau \int_D E_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(\varepsilon),$$

leading to

$$\int_D E_\varepsilon(\tau)\,\mathrm{d}x \leqslant C\int_D E_\varepsilon(0)\,\mathrm{d}x + \mathcal{O}(\varepsilon);$$

Convergence

Outcome:

$$\left[\int_D E_\varepsilon \, \mathrm{d}x\right]_{t=0}^{t=\tau} + \text{sth. non-neg.} \leqslant C \int_0^\tau \int_D E_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \mathcal{O}(\varepsilon),$$

leading to

$$\int_D E_\varepsilon(\tau) \, \mathrm{d}x \leqslant C \int_D E_\varepsilon(0) \, \mathrm{d}x + \mathcal{O}(\varepsilon);$$

hence, for any $\tau \in (0, T)$, if $\int_D E_\varepsilon(0) \, \mathrm{d}x \to 0$, then

$$\lim_{\varepsilon \to 0} \int_D E_\varepsilon(\tau) \, \mathrm{d}x = 0,$$

and

$$(\mathbf{u}_\varepsilon, \vartheta_\varepsilon, \mathbf{B}_\varepsilon) \to (\mathbf{U}, \overline{\vartheta}, \overline{\mathbf{B}}) \text{ in } L^2(0, T; W^{1,2}(D)), \quad \varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0, T; L^2(D)),$$

$$\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}, \frac{\mathbf{B}_\varepsilon - \overline{\mathbf{B}}}{\varepsilon}\right) \to (\varrho^1, \vartheta^1, \mathbf{B}^1) \text{ in } L^2((0, T) \times D)$$

Scalings

- Recall

$$\mathrm{Al} = \frac{u_c}{B_c/\sqrt{\varrho_c}} = \frac{u_c}{c_A}, \qquad \mathrm{Ma} = \frac{u_c}{\sqrt{p_c/\varrho_c}} = \frac{u_c}{c_s}, \qquad \mathrm{Fr} = \frac{u_c}{\sqrt{gL_c}}$$

Scalings

- Recall

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- Usual Boussinesq scaling: $\varepsilon_1 = \mathrm{Ma} = \frac{u_c}{c_s} (= \frac{\Delta \varrho}{\overline{\varrho}}),\ u_c = \varepsilon_1^{\frac{1}{2}} \sqrt{gL_c}$

Scalings

- Recall

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- Usual Boussinesq scaling: $\varepsilon_1 = \mathrm{Ma} = \frac{u_c}{c_s} (= \frac{\Delta\varrho}{\overline{\varrho}})$, $u_c = \varepsilon_1^{\frac{1}{2}} \sqrt{gL_c}$
- Spiegel/Weiss, and Bowker/Hughes/Kersalé: $u_c \ll c_s$ and $c_A \sim u_c$, so set $\varepsilon_2 = \frac{c_A^2}{c_s^2} \ll 1$. Then $u_c \sim \varepsilon_2^{\frac{1}{2}} c_s$, and $\varepsilon_1 = \varepsilon_2^{\frac{1}{2}} \equiv \varepsilon$; in turn

$$\mathrm{Al} = 1, \qquad \mathrm{Ma} = \varepsilon, \qquad \mathrm{Fr} = \varepsilon^{\frac{1}{2}}$$

Scalings

- Recall

$$\mathrm{Al} = \frac{u_c}{B_c/\sqrt{\varrho_c}} = \frac{u_c}{c_A}, \qquad \mathrm{Ma} = \frac{u_c}{\sqrt{p_c/\varrho_c}} = \frac{u_c}{c_s}, \qquad \mathrm{Fr} = \frac{u_c}{\sqrt{gL_c}}$$

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- Spiegel/Weiss, and Bowker/Hughes/Kersalé: $u_c \ll c_s$ and $c_A \sim u_c$, so set $\varepsilon_2 = \frac{c_A^2}{c_s^2} \ll 1$. Then $u_c \sim \varepsilon_2^{\frac{1}{2}} c_s$, and $\varepsilon_1 = \varepsilon_2^{\frac{1}{2}} \equiv \varepsilon$; in turn

$$\mathrm{Al} = 1, \qquad \mathrm{Ma} = \varepsilon, \qquad \mathrm{Fr} = \varepsilon^{\frac{1}{2}}$$

- Our case: $u_c \ll c_s$ and $c_A \sim c_s$; thus, with $\tilde{\varepsilon}_2 = \frac{u_c}{c_A} \ll 1$, get $u_c = \varepsilon_1 c_s = \tilde{\varepsilon}_2 c_A \sim \tilde{\varepsilon}_2 c_s$, hence $\varepsilon_1 = \tilde{\varepsilon}_2 \equiv \varepsilon$ and

$$\mathrm{Al} = \varepsilon, \qquad \mathrm{Ma} = \varepsilon, \qquad \mathrm{Fr} = \varepsilon^{\frac{1}{2}}.$$

"Mathematical" magneto-OB

Recall our target system:

$$
\begin{aligned}
\operatorname{div} \mathbf{U} = 0, \quad \operatorname{div} \mathbf{B} = 0, \\
\overline{\varrho}(\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U}) - \mu \nabla^2 \mathbf{U} + \nabla \Pi = \varrho \nabla G, \\
\partial_t \mathbf{B} + (\mathbf{U} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{U} - \zeta \nabla^2 \mathbf{B} = 0, \\
\overline{\varrho} c_p (\partial_t \vartheta + \mathbf{U} \cdot \nabla \vartheta) - \overline{\varrho}\overline{\vartheta}\alpha \mathbf{U} \cdot \nabla G + \overline{\vartheta}\alpha\zeta \nabla^2 (\overline{\mathbf{B}} \cdot \mathbf{B}) - \kappa \nabla^2 \vartheta \\
= \overline{\vartheta}\alpha \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \partial_t \fint_D \vartheta \, \mathrm{d}x, \\
\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})\varrho + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})\vartheta + \overline{\mathbf{B}} \cdot \mathbf{B} = \overline{\varrho} G + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \fint_D \vartheta \, \mathrm{d}x
\end{aligned}
$$

"Physical" magneto-OB

(See Spiegel/Weiss: "Magnetic Buoyancy and the Boussinesq Approximation", 1982; Bowker/Hughes/Kersalé "Incorporating velocity shear into the magneto-Boussinesq approximation", 2014):

$$\begin{aligned}
\operatorname{div} \mathbf{U} = 0, \quad \operatorname{div} \mathbf{B} = 0, \\
\overline{\varrho}(\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U}) - \mu\nabla^2\mathbf{U} + \nabla\Pi = -\varrho g \mathbf{e}_3 + (\mathbf{B}\cdot\nabla)\mathbf{B}, \\
\partial_t \mathbf{B} + (\mathbf{U}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{U} - \zeta\nabla^2\mathbf{B} = -H_\varrho^{-1} U_3 \mathbf{B}, \\
\overline{\varrho} c_p(\partial_t\vartheta + \mathbf{U}\cdot\nabla\vartheta) - (\partial_t p + \mathbf{U}\cdot\nabla p) - \kappa\nabla^2\vartheta = -U_3\beta, \\
p = R\varrho\vartheta, \quad \Pi = p + p_m = R\varrho\vartheta + \frac{1}{2}|\mathbf{B}|^2, \\
\partial_t p + \mathbf{U}\cdot\nabla p = -\overline{\varrho} g U_3 - (\partial_t p_m + \mathbf{U}\cdot\nabla p_m), \\
\partial_t p_m + \mathbf{U}\cdot\nabla p_m = \mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{U} + \zeta\nabla^2\mathbf{B}]
\end{aligned}$$

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For us: $(\mathbf{B}\cdot\nabla)\mathbf{B} = b(t,x_1,x_2)\cdot\partial_3 b(t,x_1,x_2) = 0$,
$-g\mathbf{e}_3 = \nabla[x \mapsto -gx_3] = \nabla G$, $H_\varrho^{-1} = -\frac{\mathrm{d}}{\mathrm{d}z}\log(\overline{\varrho}) = 0$,
$\beta = \overline{\vartheta}\gamma^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\log(\overline{p}\,\overline{\varrho}^{-\gamma}) = 0 \Rightarrow$ CE, ME, IE consistent!

Comparison of HE

Our HE:

$$\overline{\varrho} c_p(\partial_t \vartheta + \mathbf{U}\cdot\nabla\vartheta) - \overline{\varrho}\,\overline{\vartheta}\alpha\mathbf{U}\cdot\nabla G + \overline{\vartheta}\alpha\zeta\nabla^2(\overline{\mathbf{B}}\cdot\mathbf{B}) - \kappa\nabla^2\vartheta = \overline{\vartheta}\alpha\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\partial_t \fint_D \vartheta \,\mathrm{d}x$$

Physical HE (according to Spiegel/Weiss):

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\overline{\varrho} c_p(\partial_t \vartheta + \mathbf{U}\cdot\nabla\vartheta) - (\partial_t p + \mathbf{U}\cdot\nabla p) - \kappa\nabla^2\vartheta &= 0,\\
p = R\varrho\vartheta, \quad \Pi = p + p_m &= R\varrho\vartheta + \frac{1}{2}|\mathbf{B}|^2,\\
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Summary

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Dziękuję za uwagę!

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