Turbulence and atmospheric boundary layer
Lecture 2

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Summary of lecture 1

1. Characteristic features of turbulence
   - non predictability, non-linearity, chaotic nature, large space and time variations of all quantities describing the flow, huge range of vortex scales, strong vortex interactions, intensification of mixing
2. Dimensional analysis
   - Non-dimensional parameters: Re, Pe, Ra, Pr, Ri
3. Stability analysis
   - Kelvin-Helmholz, Raighleigh-Taylor instabilities
4. Reynolds averaging

Governing equations in the index notation

Momentum balance

$$\frac{\partial u_i}{\partial t} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{inertia} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \underbrace{\delta_{i3} b}_{buoyancy} + \underbrace{\epsilon_{ij3} f u_j}_{Coriolis}$$

Energy balance

$$\frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} = \kappa \frac{\partial^2 b}{\partial x_j \partial x_j}$$

Continuity

$$\frac{\partial u_i}{\partial x_i} = 0$$

Ensemble average operator

- Mean over infinite number of realisations

$$\overline{\Phi(\boldsymbol{x}, t)} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \Phi^{(i)}(\boldsymbol{x}, t)$$

$$\overline{\Phi(\boldsymbol{x}, t)} = \int \phi f(\phi; \boldsymbol{x}, t) \mathrm{d}\phi$$

where $f(\phi; \boldsymbol{x}, t)$ is the one-point probability density function of the variable $\Phi(\boldsymbol{x}, t)$ and ϕ is the sample space of $\Phi(\boldsymbol{x}, t)$, i.e. the space with possible values of this variable. Note that ϕ is an independent variable

Approximations of the ensemble average

- Mean over N realizations

$$\overline{\Phi(\boldsymbol{x},t)} \approx \overline{\Phi(\boldsymbol{x},t)}_N = \frac{1}{N}\sum_{i=1}^{N}\Phi^{(i)}(\boldsymbol{x},t)$$

- Time average

$$\overline{\Phi(\boldsymbol{x},t)} \approx \overline{\Phi(\boldsymbol{x},t)}_T = \frac{1}{T}\int_{t-T/2}^{t+T/2}\Phi(\boldsymbol{x},\tau)\mathrm{d}\tau,$$

  where T should be larger than characteristic time scale of turbulent motions but smaller than the time scale of mean changes.

- Space average

$$\overline{\Phi(\boldsymbol{x},t)} \approx \overline{\Phi(\boldsymbol{x},t)}_V = \frac{1}{V}\int_V \Phi(\boldsymbol{x}',t)\mathrm{d}\boldsymbol{x}',$$

  where the volume $V = L^3$, and L should be larger than characteristic length scale of turbulent motions but smaller than the length scale of mean changes.

Properties of the ensemble average operator

$$\overline{\Phi + \Psi} = \overline{\Phi} + \overline{\Psi}$$

$$\overline{\overline{\Phi}} = \overline{\Phi}, \quad \overline{\Phi \overline{\Psi}} = \overline{\Phi \Psi}$$

$$\overline{\frac{d\Phi}{ds}} = \frac{d\overline{\Phi}}{ds}$$

But

$$\overline{\Phi \Psi} \neq \overline{\Phi}\,\overline{\Psi}$$

Reynolds decomposition

$$\Phi = \overline{\Phi} + \phi'$$

where ϕ' is the fluctuation around the mean and $\overline{\phi'} = 0$.

Reynolds-averaged equations

$$u_i = \overline{u_i} + u_i', \quad b = \overline{b} + b', \quad p = \overline{p} + p'$$

Momentum balance

$$\underbrace{\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}}_{(1)} + \underbrace{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}}_{(2)} = \underbrace{-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}}_{(3)} + \underbrace{\nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j}}_{(4)} + \underbrace{\delta_{i3}\overline{b}}_{(5)} + \underbrace{\epsilon_{ij3} f \overline{u_j}}_{(6)}$$

1. rate of change following a point moving with mean velocity
2. Reynolds-stress term (a crucial difference in comparison to N-S!!!)
3. mean pressure term
4. mean viscous term
5. mean buoyancy terms
6. mean Coriolis force term

Reynolds-averaged equations

Energy balance

$$\underbrace{\frac{\partial \overline{b}}{\partial t} + \overline{u_j}\frac{\partial \overline{b}}{\partial x_j}}_{(1)} + \underbrace{\frac{\partial \overline{b' u_j'}}{\partial x_j}}_{(2)} = \underbrace{\kappa \frac{\partial^2 \overline{b}}{\partial x_j \partial x_j}}_{(3)}$$

Continuity

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0$$

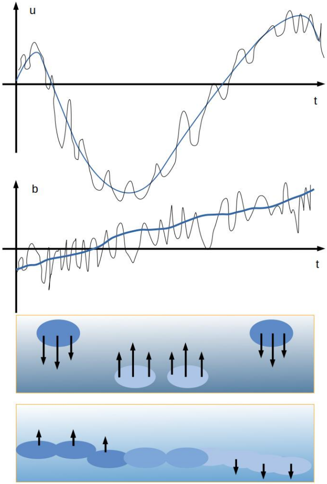
1. rate of change following a point moving with mean velocity
2. buoyancy-flux term
3. mean molecular transport term

Second order moments

Turbulent fluxes

1. Momentum flux $\overline{u_i' u_j'}$
2. Buoyancy flux $\overline{b' u_i'}$

Positive vs. negative buoyancy flux



Closure problem

Dependent variables in Navier-Stokes system:

$$u(\boldsymbol{x},t), v(\boldsymbol{x},t), w(\boldsymbol{x},t), p(\boldsymbol{x},t), b(\boldsymbol{x},t)$$

which gives 5 unknowns and 3+1+1 =5 equations

Dependent variables in Reynolds equations:

$$\overline{u(\boldsymbol{x},t)}, \overline{v(\boldsymbol{x},t)}, \overline{w(\boldsymbol{x},t)}, \overline{p(\boldsymbol{x},t)}, \overline{b(\boldsymbol{x},t)},$$

$$\overline{u^2(\boldsymbol{x},t)}, \overline{v^2(\boldsymbol{x},t)}, \overline{w^2(\boldsymbol{x},t)}, \overline{u(\boldsymbol{x},t)v(\boldsymbol{x},t)}, \overline{u(\boldsymbol{x},t)w(\boldsymbol{x},t)}, \overline{v(\boldsymbol{x},t)w(\boldsymbol{x},t)}$$

$$\overline{u(\boldsymbol{x},t)b(\boldsymbol{x},t)}, \overline{v(\boldsymbol{x},t)b(\boldsymbol{x},t)}, \overline{w(\boldsymbol{x},t)b(\boldsymbol{x},t)}$$

5 equations and 14 unknowns - Closure problem!!!

Derivation of Reynolds-stress transport equations

$$
\begin{array}{rl}
 & \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \delta_{i3} b + \epsilon_{ik3} f u_k \\
- & \frac{\partial \overline{u_i}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial \overline{u_i' u_k'}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k} + \delta_{i3}\overline{b} + \epsilon_{ik3} f \overline{u_k} \\
\hline
 & \frac{\partial u_i'}{\partial t} + \overline{u_k}\frac{\partial u_i'}{\partial x_k} + u_k' \frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial u_i' u_k'}{\partial x_k} - \frac{\partial \overline{u_i' u_k'}}{\partial x_k} = \\
 & = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} + \delta_{i3} b' + \epsilon_{ik3} f u_k'
\end{array}
$$

Procedure:

- Multiply u_i' transport equation by u_j'
- Multiply u_j' transport equation by u_i'
- Add both equations
- Average resulting equation

$$
u_j' \frac{\partial u_i'}{\partial t} + u_i' \frac{\partial u_j'}{\partial t} = \frac{\partial u_i' u_j'}{\partial t}
$$

Derivation of Reynolds-stress transport equations

RS transport equation

$$\underbrace{\frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_i' u_j'}}{\partial x_k}}_{(1)} + \underbrace{\frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{1}{\rho}\left(\frac{\partial \overline{p' u_j'}}{\partial x_i} + \frac{\partial \overline{p' u_i'}}{\partial x_j}\right)}_{(3)} = \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j}\right)}}_{(4)}$$

$$- \underbrace{\overline{u_i' u_k'}\frac{\partial \overline{u_j}}{\partial x_k} - \overline{u_i' u_k'}\frac{\partial \overline{u_j}}{\partial x_k}}_{(5)} + \underbrace{\nu \frac{\partial \overline{u_i' u_j'}}{\partial x_k \partial x_k}}_{(6)} - \underbrace{2\nu\overline{\left(\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{(7)}$$

$$+ \underbrace{\delta_{i3}\overline{b' u_j'} + \delta_{j3}\overline{b' u_i'}}_{(8)} + \underbrace{\epsilon_{ik3} f\,\overline{u_k' u_j'} + \epsilon_{jk3} f\,\overline{u_k' u_i'}}_{(9)}$$

Derivation of Reynolds-stress transport equations

1. Transport with mean velocity
2. Turbulent transport
3. Pressure transport
4. Pressure-strain rate tensor (redistribution)
5. Shear production term
6. Viscous transport
7. Dissipation tensor
8. Buoyancy production (positive or negative)
9. Coriolis term (redistribution)

References

S. B. Pope (2000)
Turbulent Flows
Cambridge University Press

References

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The End