Exercise Sheet 1

Vector and tensor identities

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1. Given a scalar field $\phi(\mathbf{r})$, a vector field $\mathbf{u}(\mathbf{r})$, and a tensor field $\mathbf{\Sigma}(\mathbf{r})$, establish the identities:[1]

   (a) $\nabla \times \nabla\phi = 0$

   (b) $\nabla \cdot (\nabla \times \mathbf{u}) = 0$

   (c) $\nabla \cdot (\phi\mathbf{u}) = \phi\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\phi$

   (d) $\nabla \times (\phi\mathbf{u}) = \phi\nabla \times \mathbf{u} + (\nabla\phi) \times \mathbf{u}$

   (e) $(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right)$

   (f) $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$

   (g) $\mathbf{\Sigma} = \nabla \cdot (\mathbf{\Sigma r}) - (\nabla \cdot \mathbf{\Sigma})\mathbf{r}$, where $\mathbf{\Sigma r} \equiv \mathbf{\Sigma} \otimes \mathbf{r}$ is a third-rank tensor and $(\nabla \cdot \mathbf{\Sigma})\mathbf{r} \equiv (\nabla \cdot \mathbf{\Sigma}) \otimes \mathbf{r}$ is a second-rank tensor.

   (h) $\mathbf{u} \cdot (\nabla \cdot \mathbf{\Sigma}) = \nabla \cdot (\mathbf{\Sigma} \cdot \mathbf{u}) - \mathbf{\Sigma} : \nabla\mathbf{u}$, where $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$ is the inner product of two second-rank tensors $\mathbf{A}$ and $\mathbf{B}$.

2. A second-rank tensor $\mathbf{W}$ is antisymmetric if,

$$\mathbf{W}^{\mathsf{T}} = -\mathbf{W}, \tag{1}$$

   where $\mathbf{W}^{\mathsf{T}}$ is the transpose of $\mathbf{W}$.

   (a) Check that an antisymmetric second-order tensor is traceless and has only three independent elements.

   (b) The *vector cross* $\mathbf{W}$ of a vector $\mathbf{w}$ is defined by

$$\mathbf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}, \tag{2}$$

   for all vectors $\mathbf{a}$. Show that $\mathbf{W}$ is antisymmetric and has components

$$W_{ij} = -\epsilon_{ijk}w_k, \tag{3}$$

[1] Throughout this text, lightface Latin and Greek letters denote scalars. Boldface lowercase Latin and Greek letters denote vectors. Boldface uppercase Latin and Greek letters denote tensors.

where ϵ_{ijk} is the Levi-Civita permutation symbol, defined by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \text{ or } \{3,1,2\}, \\ -1, & \text{if } \{i,j,k\} = \{2,1,3\}, \{1,3,2\}, \text{ or } \{3,2,1\}, \\ 0, & \text{if any index is repeated.} \end{cases} \tag{4}$$

In the expression (3) above, the vector $\mathbf{w}$ is written in an embedded form as a reducible second-rank tensor.

(c) The *axial vector* $\mathbf{w}$ of an antisymmetric tensor $\mathbf{W}$ is defined through Eq. (2). Show that $\mathbf{w}$ has components

$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk}. \tag{5}$$

Use the $\epsilon - \delta$ identity:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{6}$$

where δ_{ij} is the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{7}$$

(d) The vorticity $\boldsymbol{\omega}$ of a vector field $\mathbf{u}$ is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{8}$$

Show that $\frac{1}{2}\boldsymbol{\omega}$ is the axial vector of the antisymmetric part

$$-\frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathsf{T}}) \tag{9}$$

of velocity gradient[2] tensor $\nabla\mathbf{u}$.

[2]We define the gradient $\nabla\mathbf{u}$ of the vector field $\mathbf{u}$ in terms of the Taylor expansion of the field (assuming such an expansion exists), viz.

$$\mathbf{u}(\mathbf{r} + \delta\mathbf{r}) = \mathbf{u}(\mathbf{r}) + \delta\mathbf{r} \cdot \nabla\mathbf{u} + o(|\delta\mathbf{r}|) \quad \text{as} \quad \delta\mathbf{r} \to 0,$$

with components

$$\mathbf{e}_i \cdot \nabla\mathbf{u} = \frac{\partial\mathbf{u}}{\partial r_i} = \lim_{h\to 0}\frac{\mathbf{u}(\mathbf{r} + h\mathbf{e}_i) - \mathbf{u}(\mathbf{r})}{h},$$

and

$$(\nabla\mathbf{u})_{ij} = (\mathbf{e}_i \cdot \nabla\mathbf{u}) \cdot \mathbf{e}_j = \frac{\partial u_j}{\partial r_i}.$$

3. Every (reducible) second-rank tensor $\mathbf{A}$ may be decomposed into a sum of irreducible parts:

$$\mathbf{A} = \mathbf{A}^{(0)} + \mathbf{A}^{(1)} + \mathbf{A}^{(2)}, \tag{10}$$

with

$$A_{ij}^{(0)} = \frac{1}{3} A_{kk}\delta_{ij}, \quad A_{ij}^{(1)} = \frac{1}{2}(A_{ij} - A_{ji}), \quad A_{ij}^{(2)} = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}A_{kk}\delta_{ij}. \tag{11}$$

Note that $\mathbf{A}^{(0)}$ is a scalar (the trace of $\mathbf{A}$) in an embedded form as a second-rank isotropic tensor, $\mathbf{A}^{(1)}$ is the antisymmetric part of $\mathbf{A}$, being equivalent to an axial vector, and $\mathbf{A}^{(2)}$ is the irreducible (symmetric and traceless) second-rank tensor part of $\mathbf{A}$. [3]

Show that

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^{(0)} : \mathbf{B}^{(0)} + \mathbf{A}^{(1)} : \mathbf{B}^{(1)} + \mathbf{A}^{(2)} : \mathbf{B}^{(2)}. \tag{12}$$

4. Recall the divergence theorem for a second-rank tensor field $\mathbf{T}$

$$\int_{\partial V} \mathbf{T} \cdot \mathbf{n} \mathrm{d}a = \int_V \nabla \cdot (\mathbf{T}^{\mathsf{T}}) \mathrm{d}\mathbf{r}, \tag{13}$$

where V is a bounded region with boundary ∂V, and $\mathbf{n}$ denotes the outward unit normal to the boundary ∂V of V.

Show that

$$\int_{\partial V} (\mathbf{r} \times \boldsymbol{\Sigma}^{\mathsf{T}} \cdot \mathbf{n}) \mathrm{d}a = \int_V (\mathbf{r} \times \nabla \cdot \boldsymbol{\Sigma} - 2\boldsymbol{\sigma}) \mathrm{d}\mathbf{r}, \tag{14}$$

where $\boldsymbol{\sigma}$ is the axial vector of the antisymmetric part of $\boldsymbol{\Sigma}$. This identity will be used in the derivation of the local balance equation for the angular momentum.

[3] The quantities A_{kk}, the axial vector of $\mathbf{A}^{(1)}$, and $\mathbf{A}^{(2)}$ form spherical tensors of rank 0, 1, and 2, respectively. They transform like the spherical harmonics Y_{lm} for $l = 0$, 1, and 2. For details see e.g. Jerphagnon J., Chemla D. and Bonneville R., *Adv. Phys.* **27** 609 (1978).