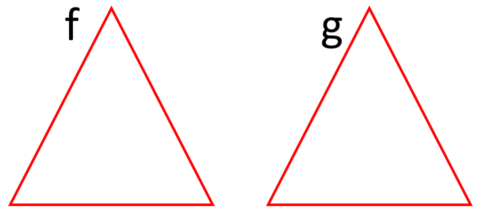
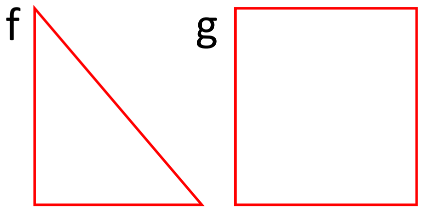
1100-4BW12, rok akademicki 2018/19

WSTĘP DO OPTYKI FOURIEROWSKIEJ

dr hab. Rafał Kasztelanic

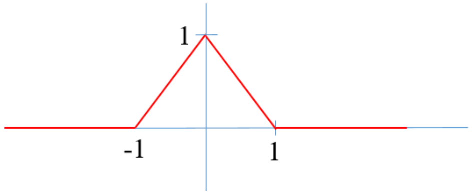
Splot





Splot

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$



$$h(x) = \int_{-\infty}^{\infty} f(x - x')\, g(x')\, \mathrm{d}x'$$

Splot

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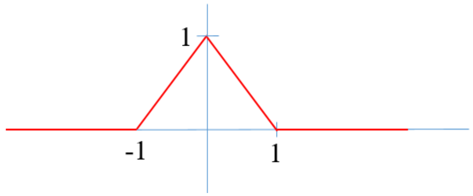

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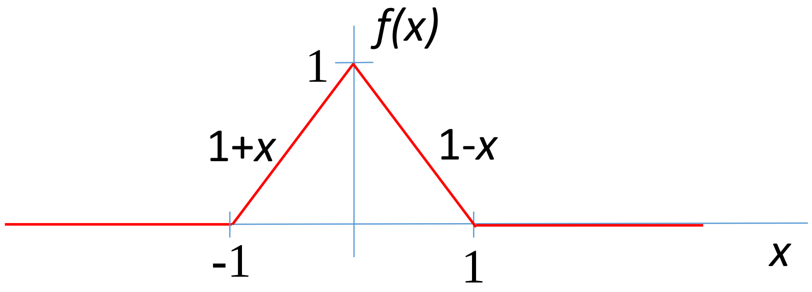

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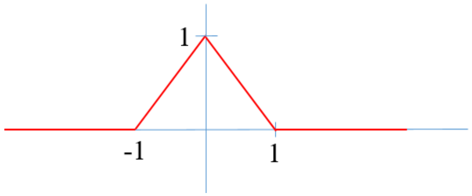

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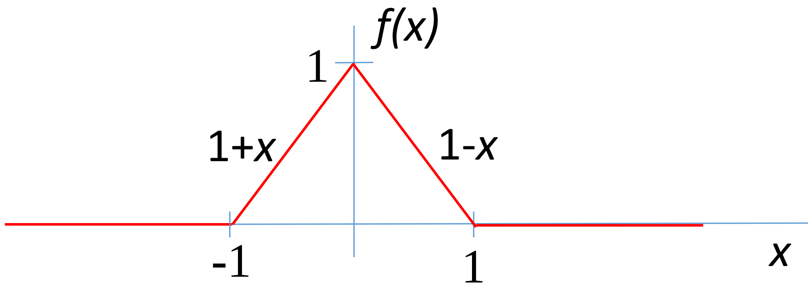

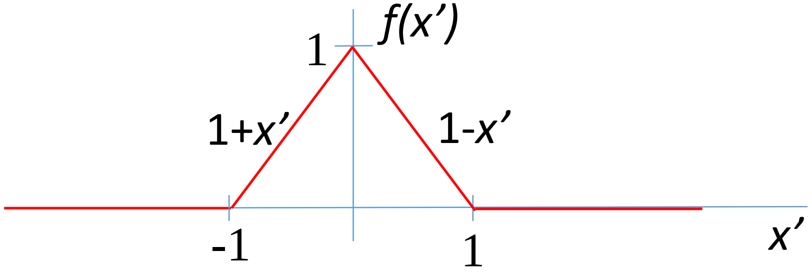

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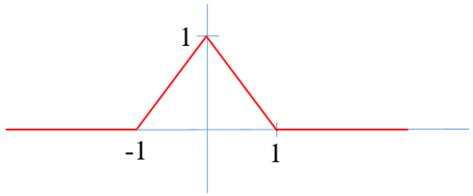

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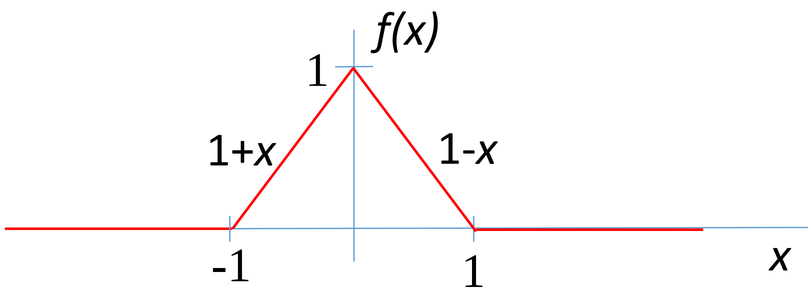

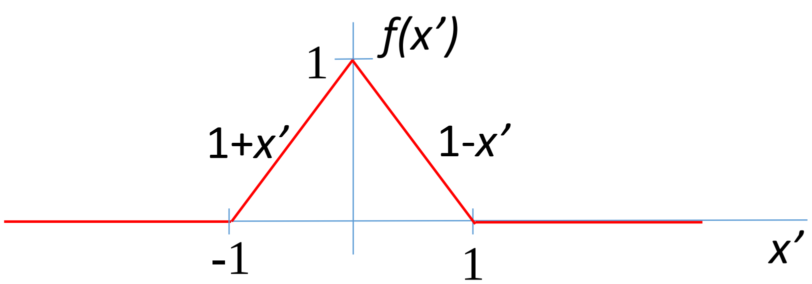

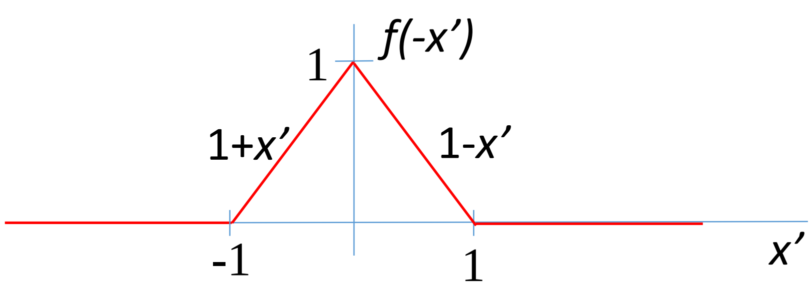

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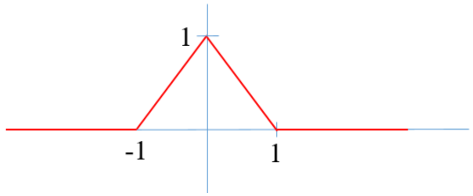
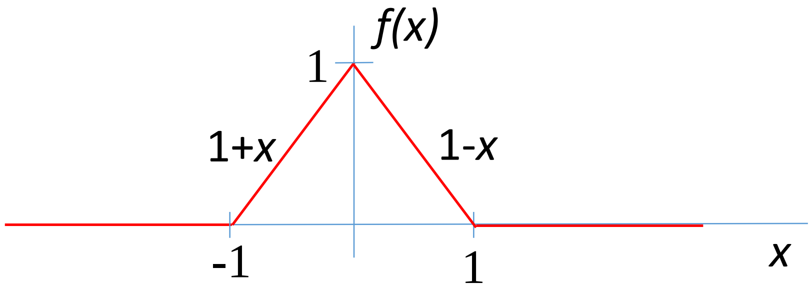
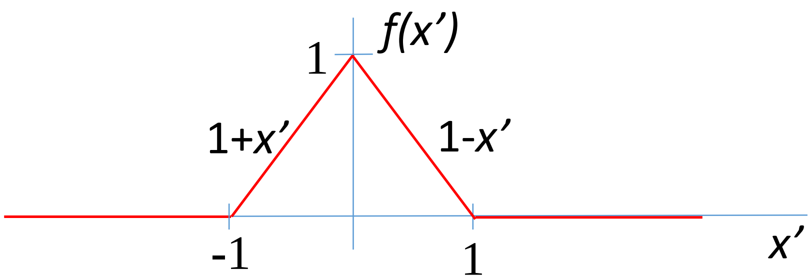
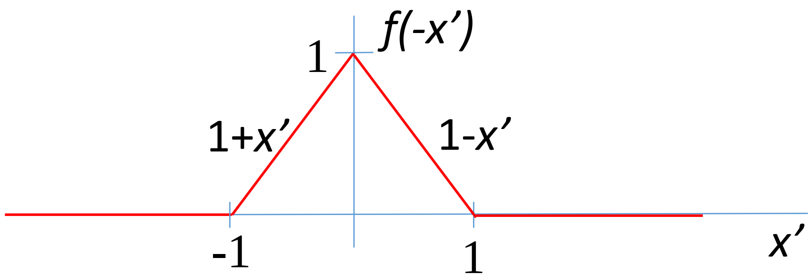
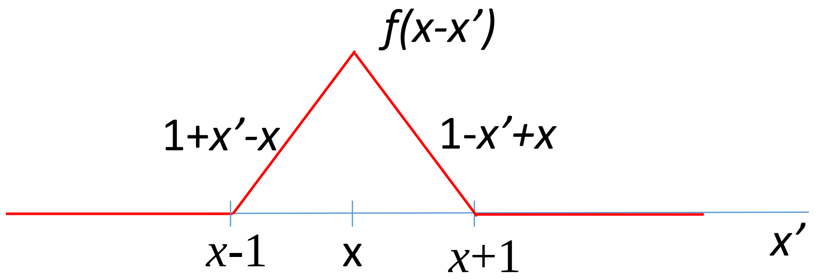


$$h(x) = \int_{-\infty}^{\infty} f(x - x') g(x') \, dx'$$



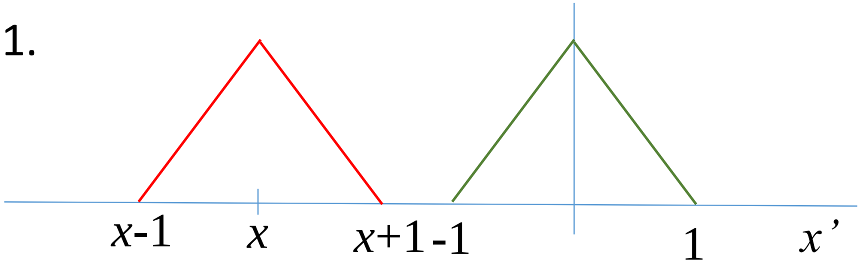






Splot

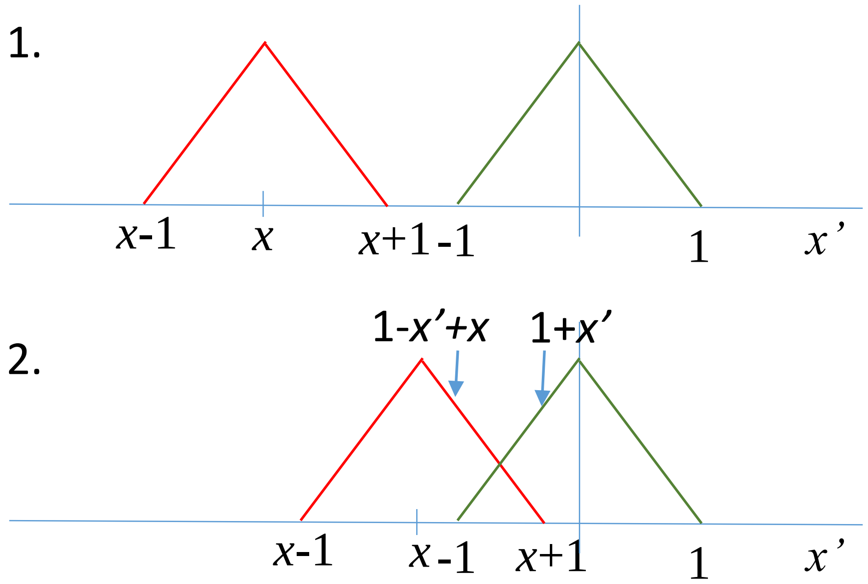
1.



$$x + 1 < -1 \rightarrow x < -2$$

$$h(x) = 0$$

Splot



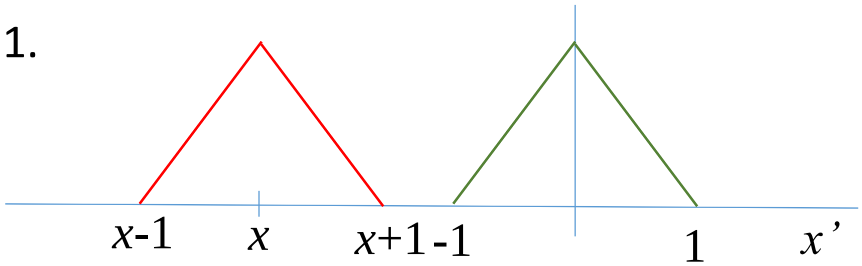
$$x + 1 < -1 \rightarrow x < -2$$

$$\boxed{h(x) = 0}$$

$$-1 < x + 1 < 0 \rightarrow -2 < x < -1$$

Splot

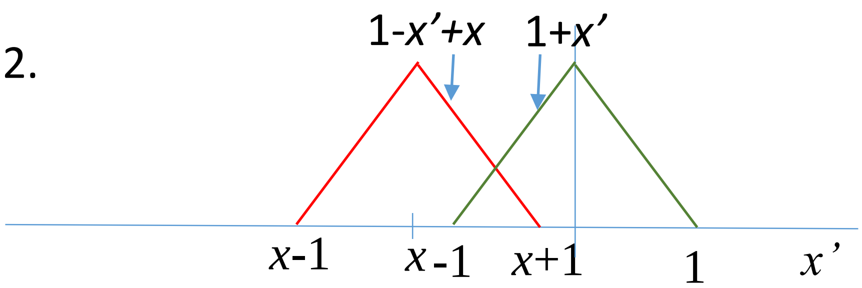
1.



$$x+1 < -1 \quad \rightarrow \quad x < -2$$

$$\boxed{h(x) = 0}$$

2.



$$-1 < x+1 < 0 \quad \rightarrow \quad -2 < x < -1$$

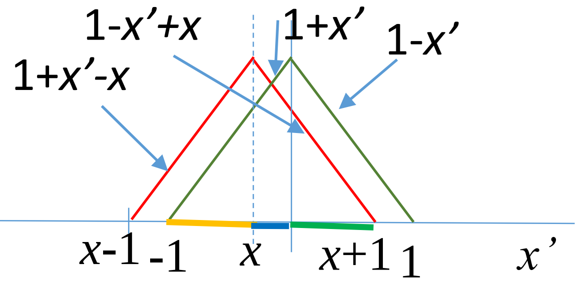
$$h(x) = \int_{-1}^{x+1} (1+x')(1-x'+x)\, dx' =$$

$$= \int_{-1}^{x+1} \left(1 + x - x'^2 + xx'\right) dx' = \underbrace{(1+x)\int_{-1}^{x+1} dx'}_{-(x+1)(x+2)} + \underbrace{x\int_{-1}^{x+1} x' dx'}_{-\frac{x}{2}\left((x+1)^2 - 1\right)} - \underbrace{\int_{-1}^{x+1} x'^2 dx'}_{-\frac{1}{3} - \frac{(x+1)^3}{3}} =$$

$$\boxed{h(x) = \frac{1}{6}(x+2)^3}$$

Splot

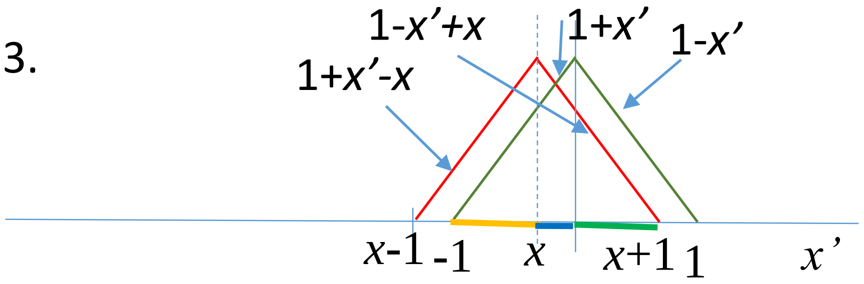
3.



$$0 < x + 1 < 1 \quad \rightarrow \quad -1 < x < 0$$

Splot

3.

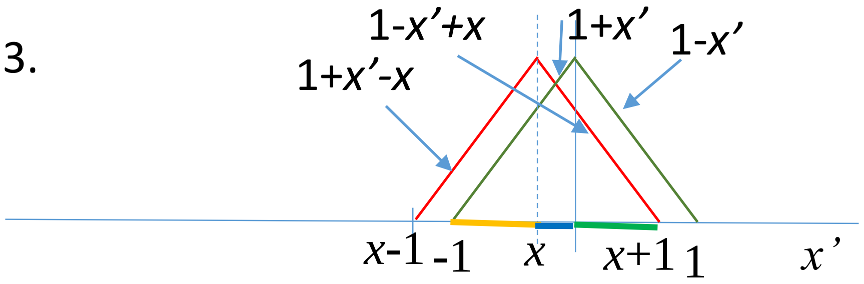


$$0 < x + 1 < 1 \;\rightarrow\; -1 < x < 0$$

$$h(x) = \underbrace{\int_{-1}^{x} (1 + x')(1 + x' - x)\, dx'}_{\frac{1}{6}(-x^3 + 3x + 2)} + \underbrace{\int_{x}^{0} (1 + x')(1 - x' + x)\, dx'}_{-\frac{1}{6}x(x^2 + 6x + 6)} + \underbrace{\int_{0}^{x+1} (1 - x')(1 - x' + x)\, dx'}_{\frac{1}{6}x(-x^3 + 3x + 2)}$$

$$\boxed{h(x) = \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6)}$$

3.



$$0 < x + 1 < 1 \;\rightarrow\; -1 < x < 0$$

$$h(x) = \int_{-1}^{x} (1 + x')(1 + x' - x)\, dx' + \int_{x}^{0} (1 + x')(1 - x' + x)\, dx' + \int_{0}^{x+1} (1 - x')(1 - x' + x)\, dx'$$

$$\frac{1}{6}(-x^3 + 3x + 2) \qquad -\frac{1}{6}x(x^2 + 6x + 6) \qquad \frac{1}{6}x(-x^3 + 3x + 2)$$

$$h(x) = \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6)$$

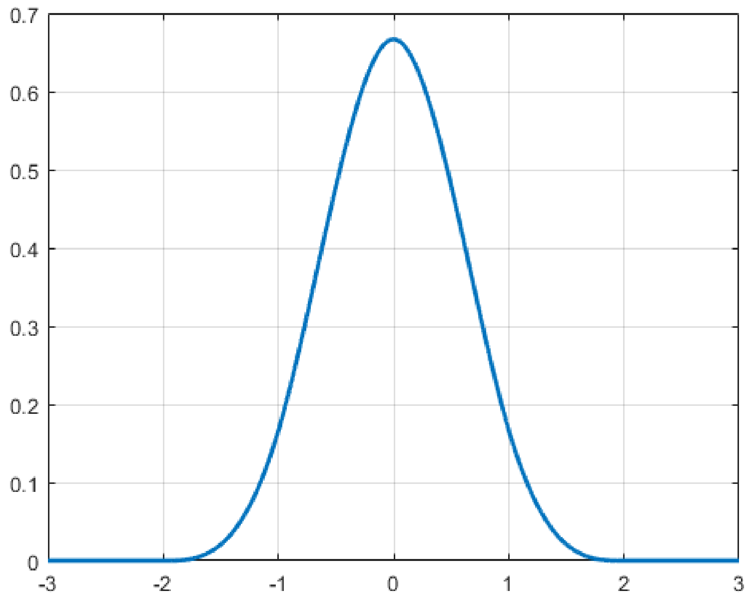
4. $-1 < x - 1 < 0 \;\rightarrow\; 0 < x < 1$ Symetrycznie do 3

5. $0 < x - 1 < 1 \;\rightarrow\; 1 < x < 2$ Symetrycznie do 2

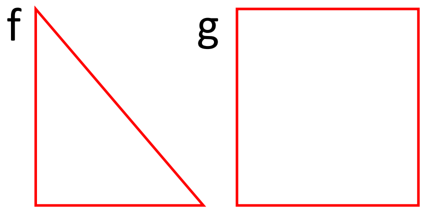
6. $1 < x - 1 \;\rightarrow\; 2 < x$ Symetrycznie do 1

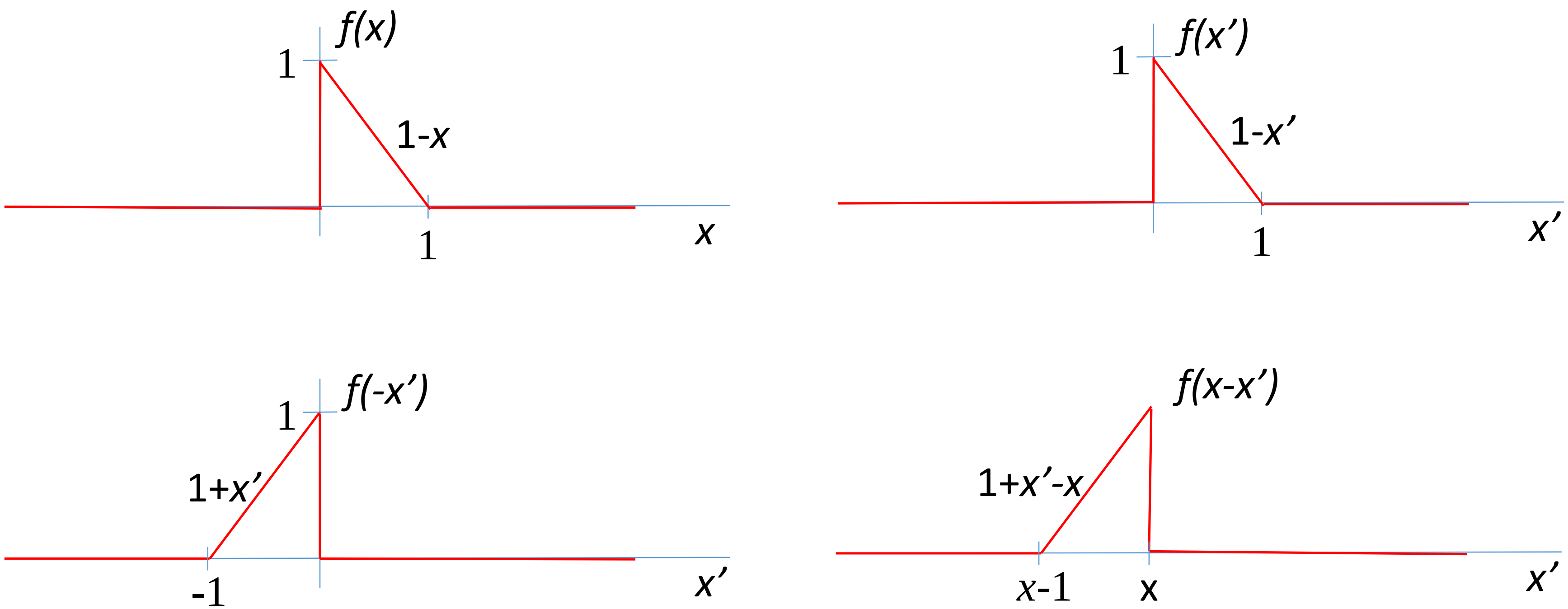
Splot

$$h(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{6}(x+2)^3 & -2 < x < -1 \\ \frac{1}{3}(-x^3 + 3x + 2) - \frac{1}{6}x(x^2 + 6x + 6) & -1 < x < 0 \\ \frac{1}{3}(x^3 - 3x + 2) + \frac{1}{6}x(x^2 - 6x + 6) & 0 < x < 1 \\ -\frac{1}{6}(x-2)^3 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$



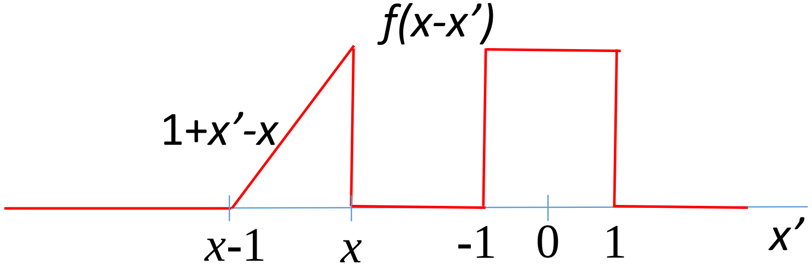
Splot





Splot

1.

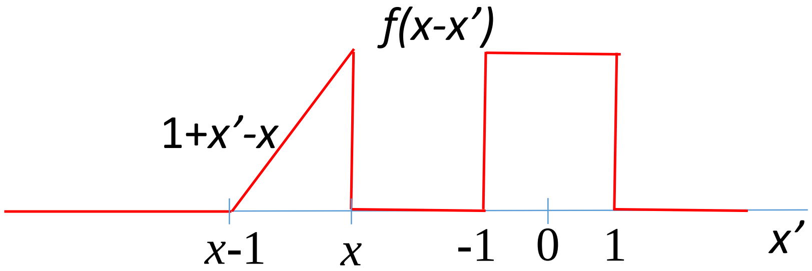


$x < -1$

$$h(x) = 0$$

Splot

1.



$$x < -1$$

$$\boxed{h(x) = 0}$$

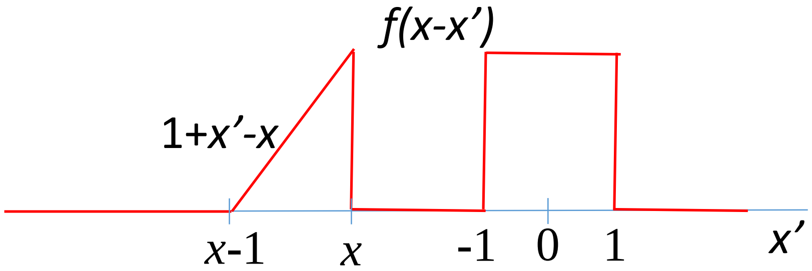
2. $-1 < x < 0$

$$h(x) = \int_{-1}^{x} (1 + x' - x)dx' = (1 - x)\int_{-1}^{x} dx' + \int_{-1}^{x} x'dx' =$$

$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \boxed{\frac{1}{2}(1 - x^2)}$$

Splot

1.



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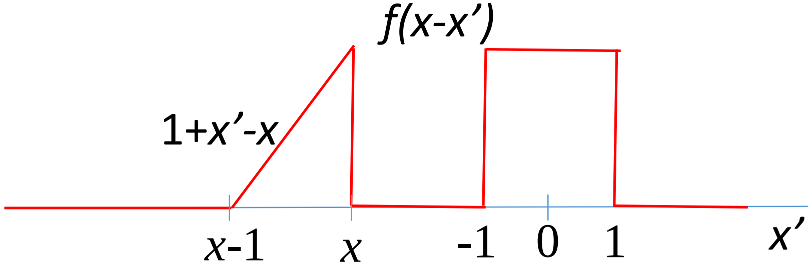
3. $0 < x < 1$

$$h(x) = \frac{1}{2}$$

Splot

1.



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$$= (1 - x)(x + 1) + \frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

3. $0 < x < 1$

$$h(x) = \frac{1}{2}$$

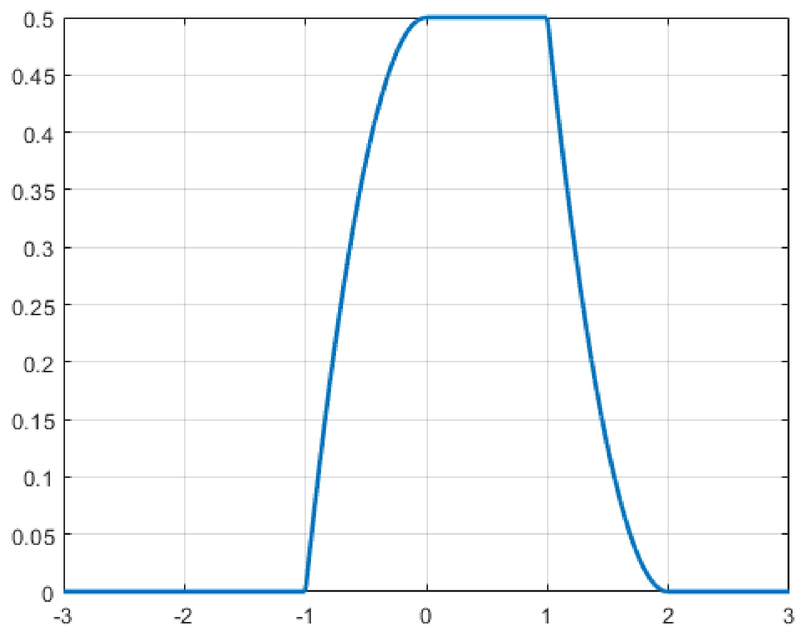
4. $0 < x - 1 < 1 \quad \rightarrow \quad 1 < x < 2$

$$h(x) = \int_{x-1}^{1} (1 + x' - x)dx' = (1 - x) \int_{x-1}^{1} dx' + \int_{x-1}^{1} x'dx' =$$

$$= (1 - x)(2 - x) + \frac{1}{2}[1 - (x - 1)^2] = \frac{x^2}{2} - 2x + 2$$

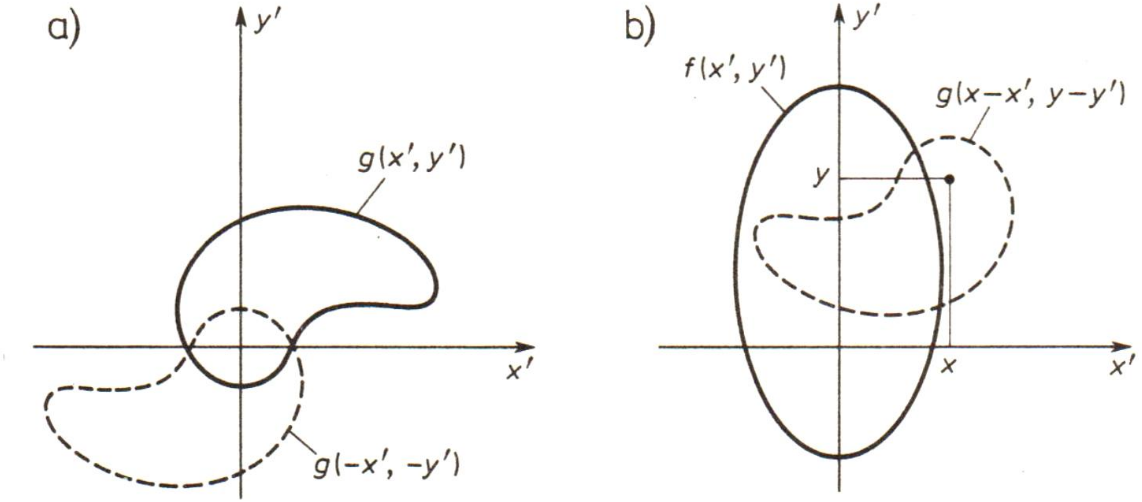
Splot

$$h(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1 - x^2) & -1 < x < 0 \\ \frac{1}{2} & 0 < x < 1 \\ \frac{x^2}{2} - 2x + 2 & 1 < x < 2 \\ 0 & 2 < x \end{cases}$$



Splot.m

Splot 2D



Korelacja

$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')\,\mathrm{d}x'$$

Korelacja

$$\varphi(x) = \int_{-\infty}^{\infty} f(x')g(x' - x)dx'$$

$$\varphi(x) = f(x) \star g(x)$$

Splot:

$$h(x) = \int_{-\infty}^{\infty} f(x - x')g(x')\,\mathrm{d}x'$$

Wasności:

Korelacja nie jest przemienna:

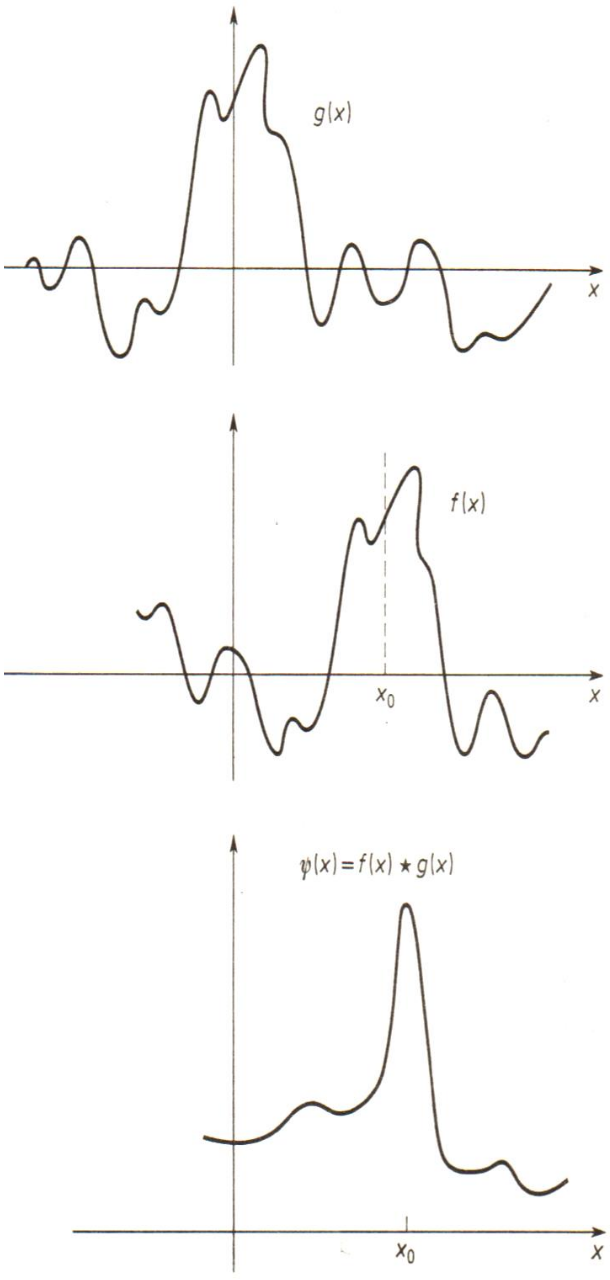
$$f(x) \star g(x) \neq g(x) \star f(x)$$

Korelacja jest równa splotowi z odwróconą funkcją *g*:

$$f(x) \star g(x) = f(x) \otimes g(-x)$$

Gdy g(x) jest funkcją parzystą to korelacja jest równoważna splotowi

Korelacja



Autokorelacja

Autokorelacja gdy $g(x) = f(x)$

Współczynnik autokorelacji: $\gamma(x) = \dfrac{\varphi(x)}{\varphi(0)}$

Moduł autokorelacji osiąga największą wartość w ‚0' : $|\varphi(x)| \leq \varphi(0)$

Transformacja Fouriera

Szereg Taylora:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

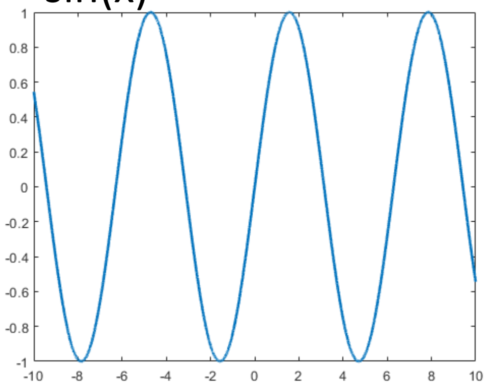
Transformacja Fouriera

Szereg Taylora:

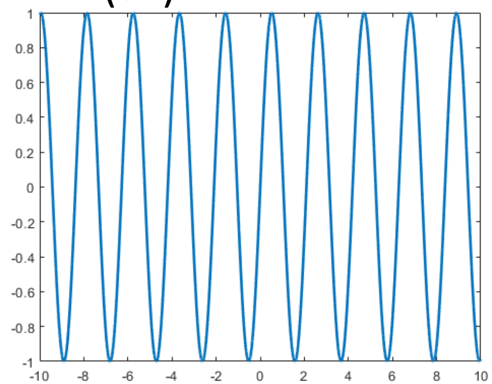
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

Suma sinusów:

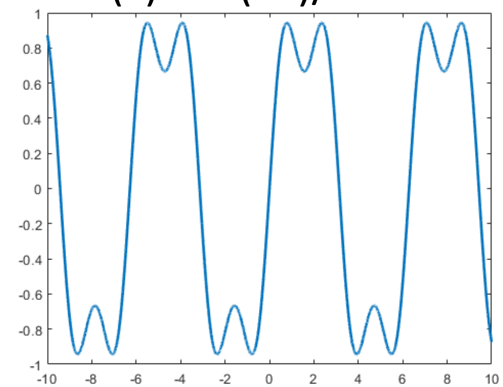
sin(x)



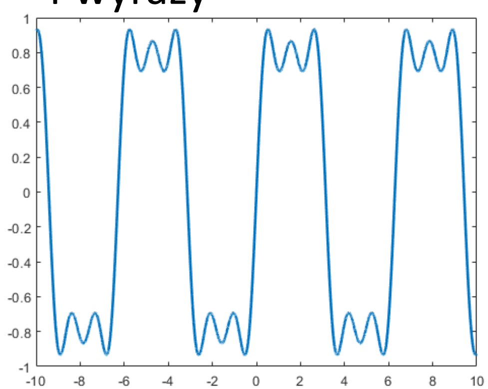
sin(3x)



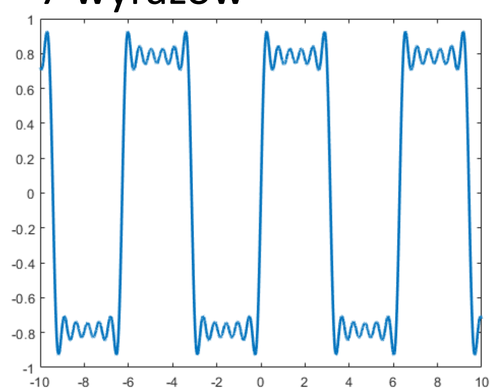
sin(x)+sin(3x)/3



4 wyrazy



7 wyrazów



$$\sum_{n=1}^{N} \frac{\sin[(2n-1)x]}{2n-1}$$

Transformacja Fouriera

Funkcję periodyczną można przedstawić jako sumę sinusów:

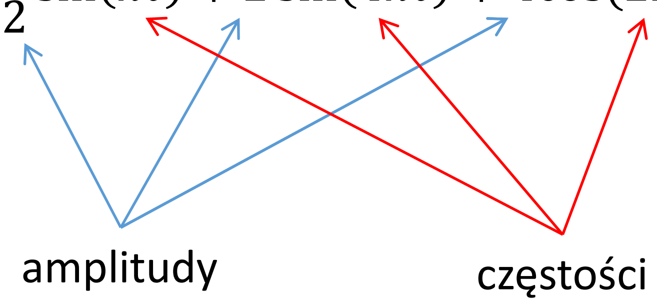
$$f(t)=\sum_{k=1}^{n}[A_k \sin(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t + \pi/2)]$$

Wygodniej to przepisać jako sumę sinusa i cosinusa:

$$f(t)=\sum_{k=1}^{n}[A_k \cos(2\pi\omega_k t) + B_k \sin(2\pi\omega_k t)]$$

Np. dla funkcji:

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$



Transformacja Fouriera

Zapis zespolony funkcji trygonometrycznych:

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) \qquad e^{-i\varphi} = \cos(\varphi) - i\sin(\varphi)$$

$$\cos(\varphi)=\frac{e^{i\varphi}+e^{-i\varphi}}{2} \qquad \sin(\varphi)=\frac{e^{i\varphi}-e^{-i\varphi}}{2}$$

Czyli $f(t)=\sum_{k=1}^{n}[A_k\cos(2\pi\omega_k t) + B_k\sin(2\pi\omega_k t)]$ mogę zapisać jako:

$$f(t)=\sum_{k=1}^{n}\left[\frac{A_k}{2}\left(e^{2\pi i\omega_k t}+e^{-2\pi i\omega_k t}\right) + \frac{B_k}{2}(e^{2\pi i\omega_k t}-e^{-2\pi i\omega_k t})\right]$$

Podstawiam:

$$C_k=\begin{cases}\frac{A_k-iB_k}{2} & dla \quad k>0\\ \frac{A_k+iB_k}{2} & dla \quad k<0\end{cases} \qquad \omega_k=\omega_{-k} \quad dla \quad k<0$$

Transformacja Fouriera

Dostaję:

f(t)=$\sum_{k=-n}^{n}[C_k e^{2\pi i \omega_k t}]$

Czyli nasza funkcja:

$$f_1 = \frac{1}{2}\sin(\pi t) + 2\sin(4\pi t) + 4\cos(2\pi t)$$

k	Częstotliwość (ω_k)	C_k
3	1	1
2	2	2i
1	1/2	i/4
0	0	0
-1	1/2	-i/4
-2	2	-2i
-3	1	-1

Transformacja Fouriera

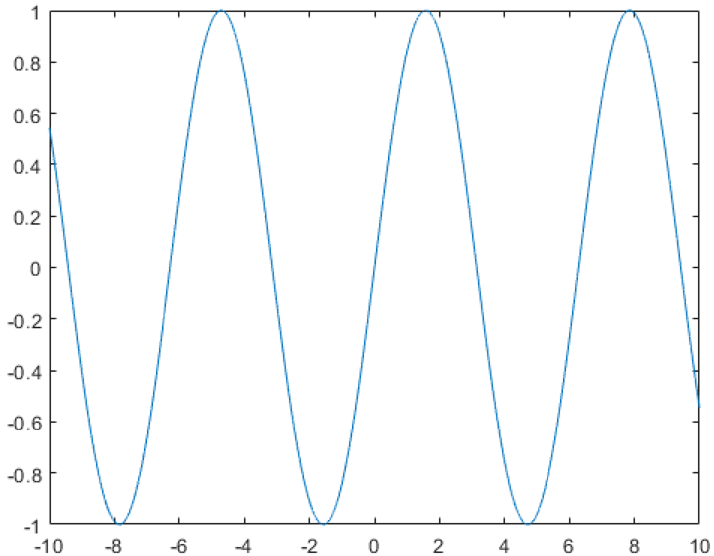
Przechodzimy do funkcji ciągłej:

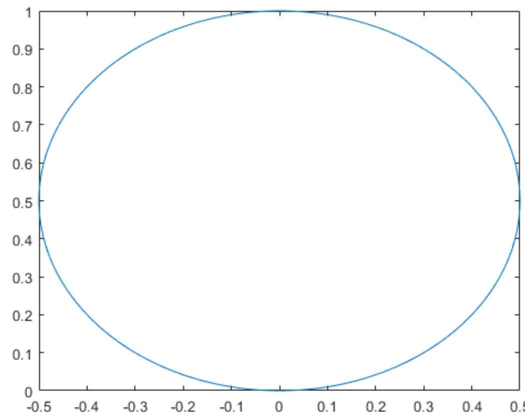
$f(\mathrm{t})=\int_{-\infty}^{\infty} F(\omega)\, e^{2\pi i\omega t}\mathrm{d}\omega$ **ODWROTNA TRANSFORMATA FOURIERA**

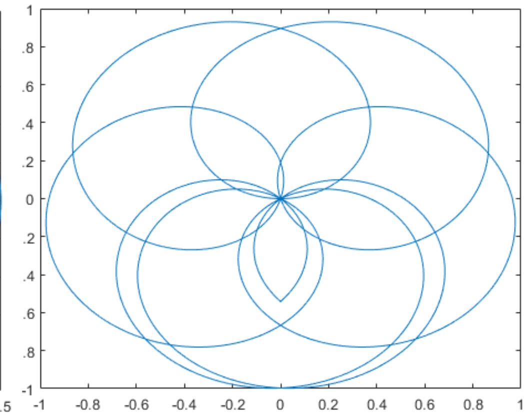
Składowe częstości ~ C_k

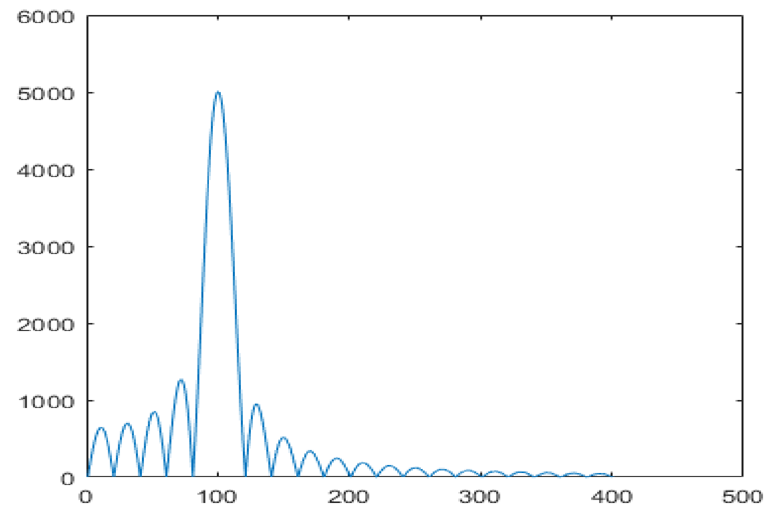
$F(\mathrm{t})=\int_{-\infty}^{\infty} f(\omega)\, e^{-2\pi i\omega t}\mathrm{d}\omega$ **TRANSFORMATA FOURIERA**

Analogia z centrum masy









FourierMasa.m

Transformacja Fouriera 2D

$$F(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} f(x, y) exp[-i2\pi(x\nu_x + y\nu_y)]\, dx\, dy$$

transformata

$$f(x, y) = \iint_{-\infty}^{\infty} F(\nu_x, \nu_y) exp[i2\pi(x\nu_x + y\nu_y)]\, d\nu_x\, d\nu_y$$

transformata odwrotna

$$f(x, y) \Leftrightarrow F(\nu_x, \nu_y)$$

Transformacja Fouriera 2D

$$f(x,y) \Leftrightarrow F(\nu_x,\nu_y) \qquad g(x,y) \Leftrightarrow G(\nu_x,\nu_y)$$

Właściwości:

Twierdzenie o liniowości:

$$\mathcal{F}\{af(x,y)+bg(x,y)\} = aF\left(\nu_x,\nu_y\right)+bG(\nu_x,\nu_y)$$

Twierdzenie o podobieństwie (skali):

$$\mathcal{F}\{f(ax,by)\} = \frac{1}{|ab|}F\left(\frac{\nu_x}{a},\frac{\nu_y}{b}\right)$$

Twierdzenie o przesunięciu w przestrzeni położeń:

$$\mathcal{F}\{f(x-x_0,y-y_0)\} = F\left(\nu_x,\nu_y\right)\exp[-i2\pi(x_0\nu_x+y_0\nu_y)]$$

Twierdzenie o przesunięciu w przestrzeni częstości:

$$\mathcal{F}\{f(x,y)\exp[i2\pi(x\nu_1+y\nu_2)\} = F\left(\nu_x-\nu_1,\nu_y-\nu_2\right)$$

Twierdzenie o wzajemności transformat:

$$\mathcal{F}\mathcal{F}\{f(x,y)\} = f(-x,-y) \qquad \mathcal{F}^{-1}\mathcal{F}\{f(x,y)\} = f(x,y)$$

Transformacja Fouriera 2D

Twierdzenie o splocie w przestrzeni położeń:

$$\mathcal{F}\{f(x,y) \otimes g(x,y)\} = F(\nu_x, \nu_y)G(\nu_x, \nu_y)$$

$$f(x,y) \star g(x,y) = \mathcal{F}^{-1}\{F(\nu_x, \nu_y)G(\nu_x, \nu_y)\}$$

Twierdzenie o splocie w przestrzeni częstości:

$$\mathcal{F}\{f(x,y)g(x,y)\} = F(\nu_x, \nu_y) \star G(\nu_x, \nu_y)$$

Twierdzenie o korelacji wzajemnej:

$$\mathcal{F}\{f(x,y) \star g^*(x,y)\} = F(\nu_x, \nu_y)G^*(\nu_x, \nu_y)$$

$$\mathcal{F}\{f(x,y) \star f^*(x,y)\} = \left|F(\nu_x, \nu_y)\right|^2$$

Twierdzenie o mocy:

$$\iint_{-\infty}^{\infty} |f(x,y)|^2 dxdy = \iint_{-\infty}^{\infty} \left|F(\nu_x, \nu_y)\right|^2 d\nu_x d\nu_y$$

Transformacja Fouriera 2D

Pary transformat:

$$f(x,y) \Leftrightarrow F(\nu_x,\nu_y)$$

$$sgn(x,y) = sgn(x)sgn(y) \quad \Leftrightarrow \quad \frac{1}{i\pi\nu_x}\frac{1}{i\pi\nu_y}$$

$$\mathrm{rect}(x,y) = \mathrm{rect}(x)\mathrm{rect}(y) \quad \Leftrightarrow \quad \mathrm{sinc}(\nu_x)\mathrm{sinc}(\nu_y) = \mathrm{sinc}(\nu_x,\nu_y)$$

$$\Lambda(x,y) = \Lambda(x)\Lambda(y) \quad \Leftrightarrow \quad sinc^2(\nu_x)sinc^2(\nu_y) = sinc^2(\nu_x,\nu_y)$$

$$\delta(x,y) \quad \Leftrightarrow \quad 1$$

$$\mathrm{comb}(x,y) = \mathrm{comb}(x)\mathrm{comb}(y) \quad \Leftrightarrow \quad \mathrm{comb}(\nu_x)\mathrm{comb}(\nu_y) = \mathrm{comb}(\nu_x,\nu_y)$$

$$\exp[i\pi(x+y)] \quad \Leftrightarrow \quad \delta\left(\nu_x - \frac{1}{2}, \nu_y - \frac{1}{2}\right)$$

$$\exp[-\pi(x^2+y^2)] \quad \Leftrightarrow \quad \exp[-\pi({\nu_x}^2+{\nu_y}^2)]$$

$$\mathrm{circ}(\mathrm{r}) = \mathrm{circ}(\sqrt{x^2+y^2}) \quad \Leftrightarrow \quad \frac{J_1(2\pi\rho)}{\rho} = \frac{J_1(2\pi\sqrt{{\nu_x}^2+{\nu_y}^2})}{\sqrt{{\nu_x}^2+{\nu_y}^2}}$$

Fourier.m