CONDENSATIONAL GROWTH OF CLOUD DROPLETS



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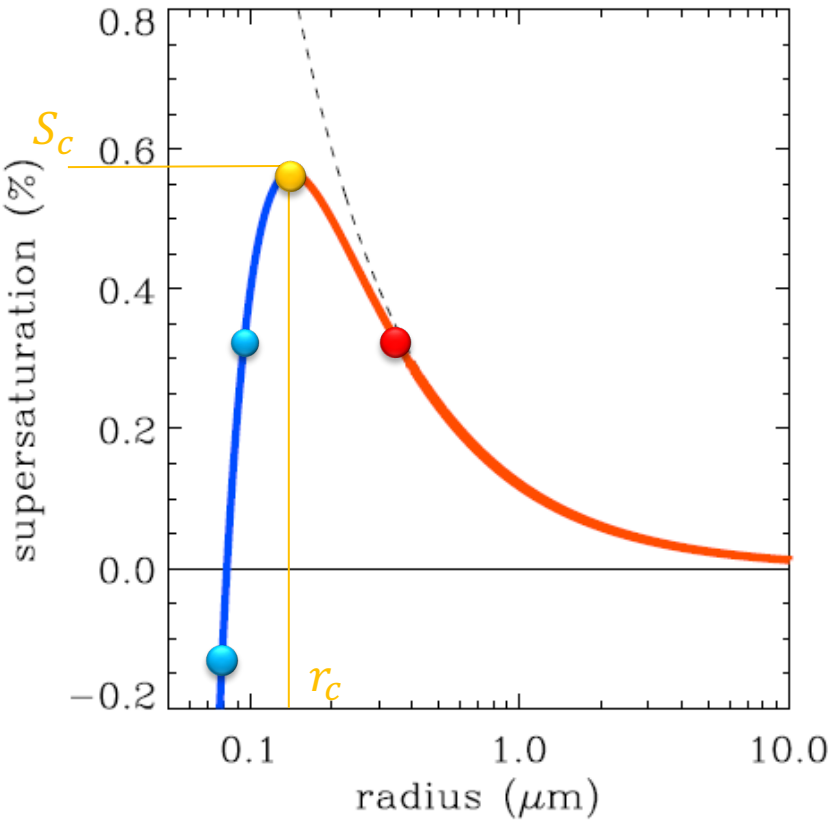
Equilibrium conditions

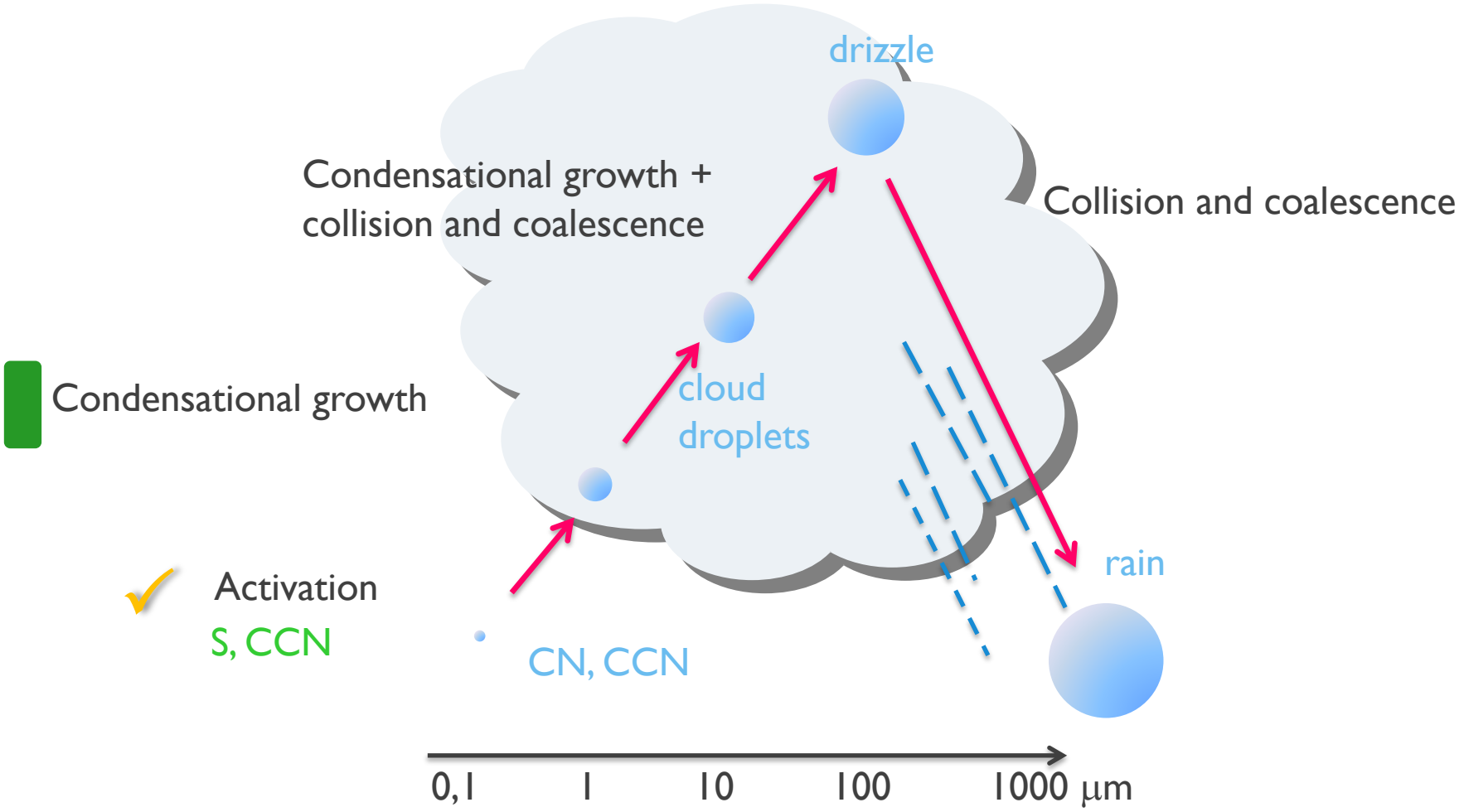


Stable equilibrium ($r < r_c$):

- An increase or decrease in ambient supersaturation causes unactivated haze particles to grow or evaporate until a new equilibrium radius is established.
- For these droplets to continue growing toward activation, ambient supersaturation must rise to the critical supersaturation (S_c) at which point the droplet reaches its critical radius (r_c).

Unstable equilibrium ($r > r_c$):

- Once ambient supersaturation exceeds the critical threshold (S_c), the droplet transitions into the unstable regime.
- In this state, droplet growth lowers the droplet's required equilibrium saturation ratio below the ambient level.
- Water vapor diffuses continuously toward the droplet, fueling spontaneous growth without requiring further increases in ambient supersaturation.


drizzle
Condensational growth +
collision and coalescence
Collision and coalescence
Condensational growth
cloud
droplets
Activation
S, CCN
rain
CN, CCN
0,1
1
10
100
1000 μm

Droplet Growth by Water Vapor Diffusion

- Solution droplets must reach the critical radius (r_c) and critical supersaturation (S_c) to activate into cloud droplets.
- Water vapor diffusion toward the droplet surface governs growth both prior to activation (the stable haze regime) and after activation (the unstable growth regime).
- Next, we will derive the condensational growth rate of an isolated droplet driven by water vapor diffusion.

Condensational growth



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Consider, an isolated spherical droplet of radius r suspended in a water vapor field;

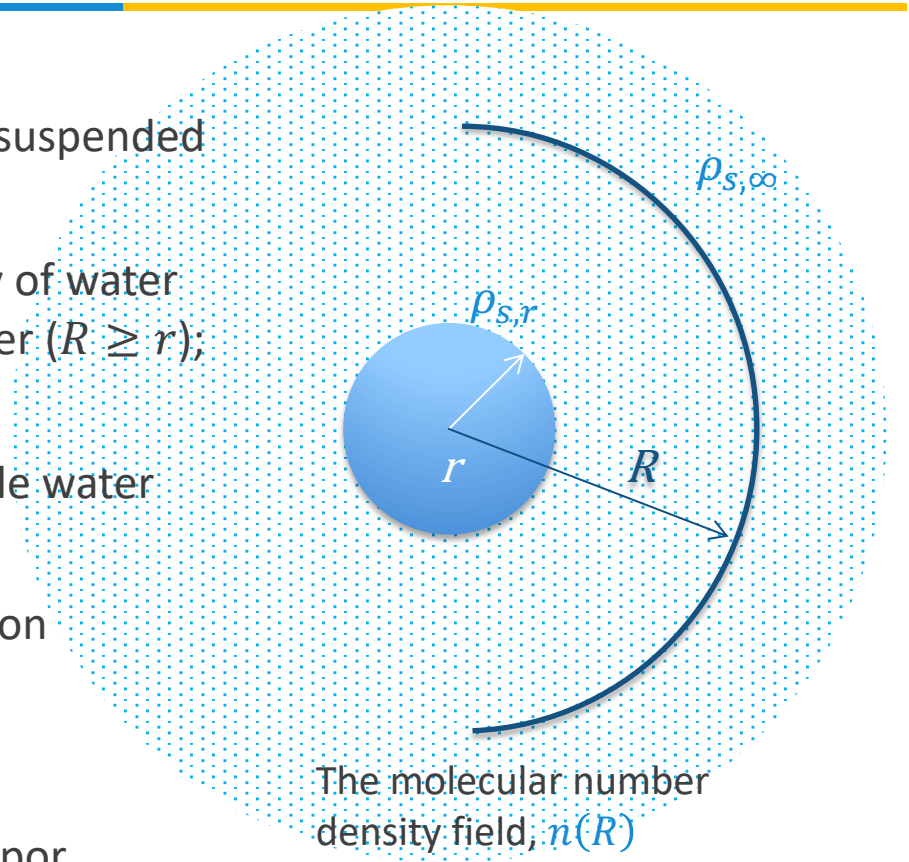
Let $n(R)$ represent the molecular number density of water vapor at a radial distance R from the droplet center ($R \geq r$);

Vapor density)or absolute humidity):
$\rho_v(R) = n(R)m_0$; where $m_0$is the mass of a single water molecule;

The vapor density is assumed to satisfy the diffusion equation:

$$\frac{\partial \rho_v}{\partial t} = D\nabla^2 \rho_v$$

D $[m^2 s^{-1}]$ is the molecular diffusivity of water vapor.



The molecular number density field, $n(R)$

The diffusion equation can be trivially derived from the continuity equation, which states that a change in density within any region of the system is due solely to the inflow and outflow of material across its boundaries. In other words, no material is created or destroyed:

$$\frac{\partial n}{\partial t} + \nabla \cdot \vec{j} = 0$$

where $\vec{j}$ is the flux of the diffusing material. According to Fick's first law of diffusion, this flux is proportional to the local density gradient:

$$\vec{j} = -D\nabla n$$

We assume steady-state (or stationary) conditions:

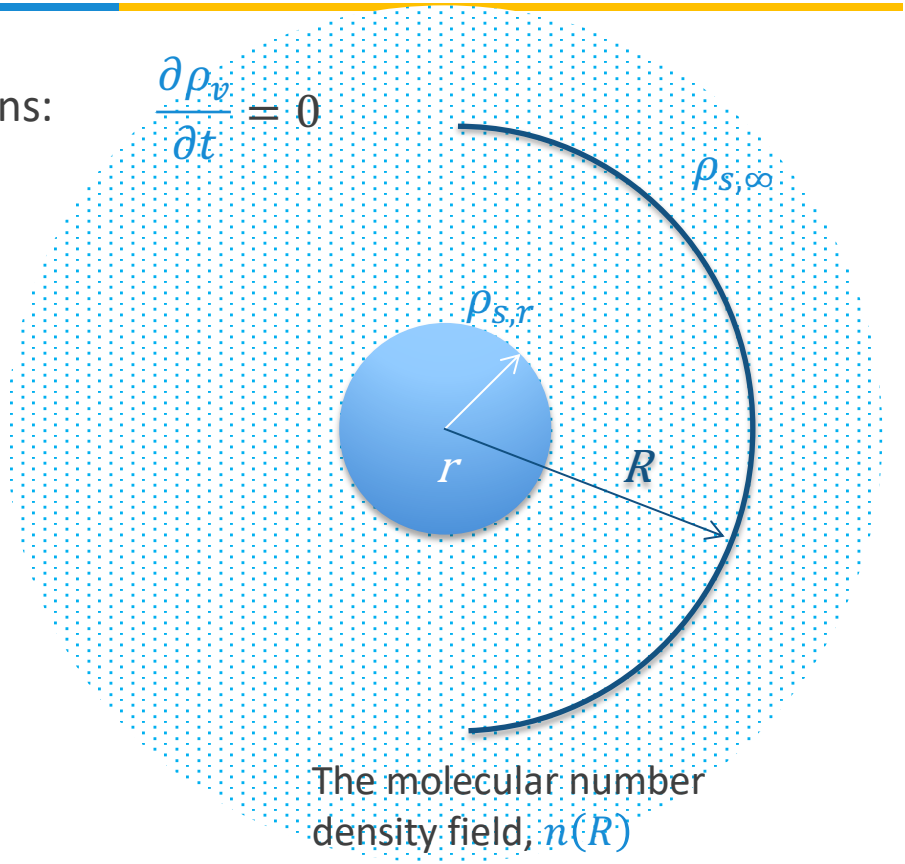
$$\frac{\partial \rho_v}{\partial t} = 0$$

$$\rho_v(R) = n(R) m_0$$

$$\frac{\partial n}{\partial t} = D\nabla^2 n \qquad \longrightarrow \qquad D\nabla^2 n = 0$$

$$D\nabla^2(R) = 0 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial n}{\partial R}\right)$$

$$n(R) = C_1 - \frac{C_2}{R}$$



The molecular number density field, $n(R)$

Boundary conditions

$R \to \infty, \quad n \to n_\infty$ The ambient (or undisturbed) water vapor concentration.

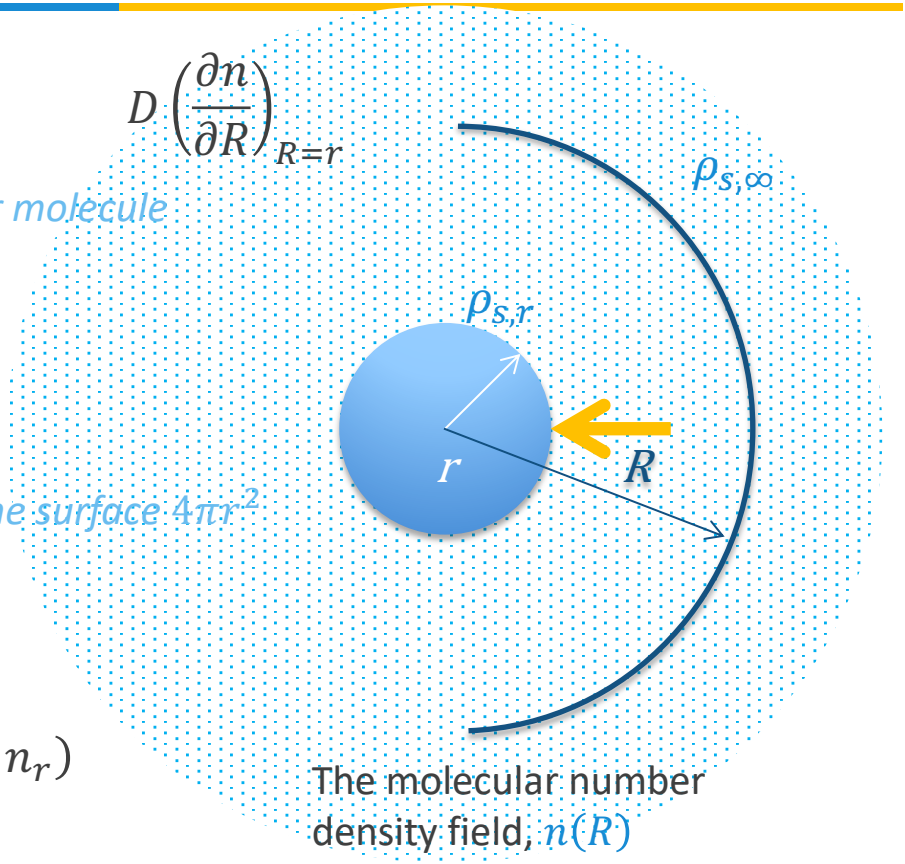
$R \to r, \quad n \to n_r$ The water vapor concentration at the droplet surface.

$$n(R) = n_\infty - \frac{r}{R}(n_\infty - n_r)$$

The radial profile of water vapor concentration around the droplet.

The flux of molecules onto the droplet surface:

$$D\left(\frac{\partial n}{\partial R}\right)_{R=r}$$



the mass of one water molecule

$$\frac{dm}{dt} = 4\pi r^2 D\left(\frac{\partial n}{\partial R}\right)_{R=r} m_0$$

the rate of mass increase

the flux of mass over the surface $4\pi r^2$

$$n(R) = n_\infty - \frac{r}{R}(n_\infty - n_r)$$

$$\left(\frac{\partial n}{\partial R}\right)_{R=r} = \frac{r}{R^2}(n_\infty - n_r)\Big|_{R=r} = \frac{1}{r}(n_\infty - n_r)$$

The diffusional growth equation for an isolated droplet at rest in a water vapor field:

$$\frac{dm}{dt} = 4\pi r D(n_\infty - n_r)m_0$$

$$\rho_v = nm_0$$

$$\frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr})$$

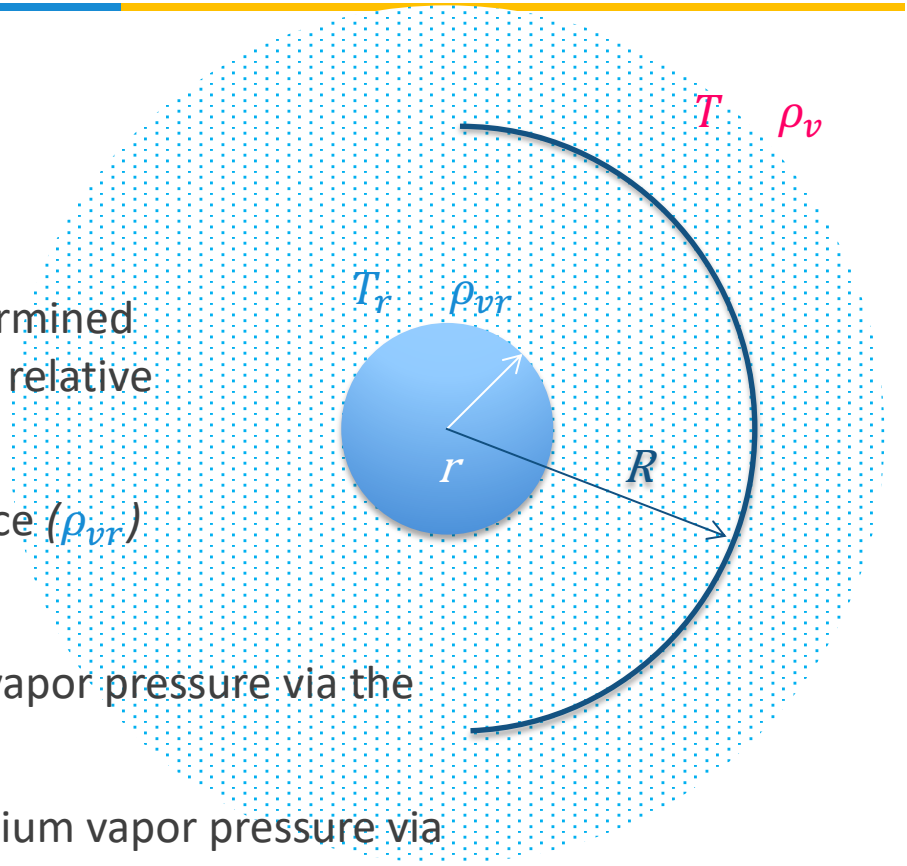
The droplet grows if $\rho_v > \rho_{vr}$

The droplet evaporates if $\rho_v < \rho_{vr}$

the ambient water vapor density

the water vapor density at the droplet surface

$$\frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr})$$



- The far-field water vapor density (ρ_v) is determined directly by the ambient temperature (T) and relative humidity (RH)
- The water vapor density at the droplet surface (ρ_{vr}) depends on:
  - **Droplet radius (r)**: Increases equilibrium vapor pressure via the curvature (Kelvin) effect.
  - **Chemical composition**: Decreases equilibrium vapor pressure via the solute (Raoult) effect.
  - **Surface temperature (T_r)**: Alters the saturation vapor density, directly affecting (ρ_{vr}).

The droplet surface temperature (T_r) generally differs from the ambient temperature (T) due to the release or absorption of latent heat during condensation or evaporation. Consequently, T_r cannot be treated as a free parameter; instead, it must be determined by solving the coupled heat and mass transfer equations for the droplet and its surrounding environment.

Latent heat released during condensation elevates the droplet surface temperature above the ambient air temperature ($T_r > T$).

Heat transfer away from the droplet is governed by Fourier's law, in direct analogy to steady-state water vapor diffusion:

$$\frac{dQ}{dt} = 4\pi r K (T_r - T)$$

The signs on the right-hand side are reversed relative to the mass diffusion equation because outward heat flux and inward mass flux are both defined as positive.

Q is the thermal energy

coefficient of thermal conductivity of air [J m^{-1} s^{-1} K^{-1}]

temperature at the droplet surface

ambient temparature

The mass and heat flux equations were first derived by James Clerk Maxwell in 1877 (in a slightly different form) in an *Encyclopaedia Britannica* article on the wet-bulb thermometer (Maxwell, 1890).

Consequently, the theory of steady-state growth for a stationary spherical droplet in a continuous water vapor field is commonly referred to as Maxwellian growth theory.

During condensational growth, the net change in cloud droplet surface temperature (T_r) is governed by the thermal energy balance between two competing processes:

- Latent heat release.
- Sensible heat dissipation.

$$mcdT_r = Ldm - dQ$$

$$\frac{4}{3}\pi r^3 \rho_l c \frac{dT_r}{dt} = L\frac{dm}{dt} - \frac{dQ}{dt}$$

water density — *specific heat capacity of water*

Assuming a steady state: $\frac{dT_r}{dt} = 0$

$$L\frac{dm}{dt} = \frac{dQ}{dt} \quad \rightarrow \quad 4\pi r D L(\rho_v - \rho_{vr}) = 4\pi r K(T_r - T)$$

$$\begin{cases} \frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr}) \\ \frac{dQ}{dt} = 4\pi r K(T_r - T) \end{cases}$$

$$\frac{\rho_v - \rho_{vr}}{T_r - T} = \frac{K}{LD}$$

Weak sensitivity to temperature and pressure; Table 7.1 R&Y

Neither the droplet surface temperature (T_r) nor the surface water vapor density (ρ_{vr}) is known *a priori.*

However, T_r and ρ_{vr} are uniquely linked through the Clausius-Clapeyron equation and Köhler theory. :

$$\rho_{vr} = \frac{e_s(r)}{R_v T_r} = \frac{\left(1 + \frac{A}{r} - \frac{\kappa r_d^3}{r^3}\right) e_s(T_r)}{R_v T_r}$$

the equilibrium vapor pressure over a plane pure-water surface at temperature T_r is given by the Clausius-Clapeyron equation

Equations:

$$\rho_{vr} = \frac{\left(1 + \frac{A}{r} - \frac{\kappa r_d^3}{r^3}\right) e_s(T_r)}{R_v T_r} \qquad \frac{\rho_v - \rho_{vr}}{T_r - T} = \frac{K}{LD}$$

constitute a system of simultaneous equations that can be solved numerically for ρ_{vr} and T_r to determine the condensational droplet growth rate:

$$\frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr})$$

Condensational growth
Mason's solution



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Analytical approximation (Mason, 1971)

As an alternative to iterative numerical solutions, Mason (1971) introduced an analytical approximation that provides an explicit, closed-form expression for the condensational growth rate of a droplet.

Assuming a saturated environment, variations in saturation vapor density (ρ_{vs}) with respect to temperature are described by the Clausius-Clapeyron equation:

$$\rho_{vs} = \frac{e_s}{R_v T} \quad \rightarrow \quad \frac{d\rho_{vs}}{\rho_{vs}} = \frac{de_s}{e_s} - \frac{dT}{T}$$

$$\frac{de_s}{e_s} = \frac{L dT}{R_v T^2}$$

Clausius-Clapeyron equation

$$\frac{d\rho_{vs}}{\rho_{vs}} = \frac{L}{R_v}\frac{dT}{T^2} - \frac{dT}{T}$$

We integrate this equation between the droplet surface temperature (T_r) and the ambient temperature (T), assuming $T/T_r \sim 1$.

Mason, B. J., 1971. The Physics of Clouds. Clarendon Press, Oxford, 671 pp.

$$\int_{T_r}^{T} \frac{d\rho_{vs}}{\rho_{vs}} = \int_{T_r}^{T} \left(\frac{L}{R_v} \frac{dT}{T^2} - \frac{dT}{T} \right)$$

$$\ln \frac{\rho_{vs}}{\rho_{vsr}} = -\frac{L}{R_v} \frac{1}{T} \Big|_{T_r}^{T} - \ln T \Big|_{T_r}^{T}$$

$$\ln \frac{\rho_{vs}}{\rho_{vsr}} = -\frac{L}{R_v} \left(\frac{1}{T} - \frac{1}{T_r} \right) - \ln \frac{T}{T_r}$$

$$\frac{T}{T_r} \approx 1 \; ; \quad \frac{T}{T_r} = 1 + \varepsilon$$

$$\ln(1+\varepsilon) \approx \varepsilon \; ; \; \ln \frac{T}{T_r} = \frac{T}{T_r} - 1$$

$$\ln \frac{\rho_{vs}}{\rho_{vsr}} = -\frac{L}{R_v} \frac{T_r - T}{T T_r} - \left(\frac{T}{T_r} - 1 \right) = (T_r - T) \left[\frac{L}{R_v T T_r} - \frac{T_r - T}{T_r (T_r - T)} \right]$$

$$\ln \frac{\rho_{vs}}{\rho_{vsr}} = (T_r - T) \left[\frac{L}{R_v T T_r} - \frac{1}{T_r} \right]$$

$$\ln\frac{\rho_{vs}}{\rho_{vsr}} = (T_r - T)\left[\frac{L}{R_v T T_r} - \frac{1}{T_r}\right]$$

$$\frac{\rho_{vs}}{\rho_{vsr}} \approx 1\,; \quad \ln\frac{\rho_{vs}}{\rho_{vsr}} \approx \frac{\rho_{vs}}{\rho_{vsr}} - 1 = \frac{\rho_{vs} - \rho_{vsr}}{\rho_{vsr}} \qquad T \cdot T_r \approx T^2$$

$$\frac{\rho_{vs} - \rho_{vsr}}{\rho_{vsr}} = \left(\frac{T - T_r}{T}\right)\left(\frac{L}{R_v T} - 1\right)$$

$T - T_r$ can be substituted from the heat transfer equation

$$\frac{dQ}{dt} = 4\pi r K (T_r - T) \longrightarrow (T_r - T) = \frac{1}{4\pi r K}\frac{dQ}{dt}$$

for a steady-state $\frac{dQ}{dt} = L\frac{dm}{dt}$

$$(T_r - T) = \frac{L}{4\pi r K}\frac{dm}{dt}$$

$$\frac{\rho_{vs} - \rho_{vsr}}{\rho_{vsr}} = \left(1 - \frac{L}{R_v T}\right)\frac{L}{4\pi r K T}\frac{dm}{dt}$$

$$\frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr}) \longrightarrow (1)\quad \frac{\rho_v - \rho_{vr}}{\rho_{vr}} = (4\pi r D \rho_{vr})^{-1} \frac{dm}{dt}$$

$$(2)\quad \frac{\rho_{vs} - \rho_{vsr}}{\rho_{vsr}} = \left(1 - \frac{L}{R_v T}\right) \frac{L}{4\pi r K T} \frac{dm}{dt}$$

Subtracting (2) from (1), assuming $\rho_{vr}=\rho_{vsr}$ and neglecting the curvature effect:

$$\frac{\rho_v - \rho_{vr}}{\rho_{vr}} - \frac{\rho_{vs} - \rho_{vsr}}{\rho_{vsr}} \approx \frac{\rho_v - \rho_{vs}}{\rho_{vsr}} \approx \frac{\rho_v}{\rho_{vs}} - 1 \approx S - 1$$

saturation over plane water surface

(1)-(2)
$$S - 1 = \frac{dm}{dt}\left[\frac{1}{4\pi r D \rho_{vr}} + \left(\frac{L}{R_v T} - 1\right)\frac{L}{4\pi r K T}\right]$$

$$\frac{dm}{dt} = \frac{S - 1}{\left[\frac{1}{4\pi r D \rho_{vr}} + \left(\frac{L}{R_v T} - 1\right)\frac{L}{4\pi r K T}\right]}$$

$$\frac{dm}{dt} = \frac{S-1}{\left[\frac{1}{4\pi r D \rho_{vr}} + \left(\frac{L}{R_v T} - 1\right)\frac{L}{4\pi r K T}\right]}$$

$$\frac{dm}{dt} = \frac{d\left(\frac{4}{3}\pi\rho_l r^3\right)}{dt} = 4\pi\rho_l r^2 \frac{dr}{dt} = (4\pi\rho_l r) r \frac{dr}{dt}$$

$$r\frac{dr}{dt} = \frac{S-1}{\left[\frac{4\pi\rho_l r}{4\pi r D \rho_{vr}} + \left(\frac{L}{R_v T} - 1\right)\frac{4\pi\rho_l r L}{4\pi r K T}\right]}$$

$$\rho_{vr} \approx \frac{e_s(T)}{R_v T}$$

$$r\frac{dr}{dt} = \frac{S-1}{\left[\frac{\rho_l R_v T}{D e_s(T)} + \left(\frac{L}{R_v T} - 1\right)\frac{\rho_l L}{K T}\right]}$$

$$r\frac{dr}{dt} = \frac{S-1}{[F_d + F_k]} \qquad F_d = \frac{\rho_l R_v T}{D e_s(T)} \qquad F_k = \left(\frac{L}{R_v T} - 1\right)\frac{\rho_l L}{K T}$$

the term associated with vapor diffusion

the term associated with heat conduction

The denominator F_d+F_k depends on both ambient temperature and pressure. However, its sensitivity to temperature is substantially more pronounced.

$$r\frac{dr}{dt} = \frac{S-1}{[F_d + F_k]}$$

saturation over a flat, pure water surface

To account for microphysical boundary corrections at the droplet surface, the equilibrium saturation ratio over a flat, pure-water surface (1) is replaced by that over a curved solution droplet (S_{eq}).

$$r\frac{dr}{dt} = \frac{S - S_{eq}}{[F_d + F_k]}$$

$$S_{eq} = 1 + \frac{A}{r} - \frac{\kappa r_d^3}{r^3}$$

$$S_{eq} = \frac{r^3 - r_d^3}{r^3 - (1-\kappa)r_d^3}\exp\left(\frac{A(T)}{r}\right)$$

This closed-form analytical equation provides a highly accurate approximation to the solution obtained by solving the coupled mass diffusion and thermal conduction equations numerically for ρ_{vr} and T_r.

In the F_k term, since $L/R_vT >> 1$, the 1 in parentheses can be neglected.

$$F_d = \frac{\rho_l R_v T}{D e_s(T)}$$

$$F_k = \left(\frac{L}{R_v T} - 1\right)\frac{\rho_l L}{KT} \approx \frac{\rho_l L^2}{K R_v T^2}$$

$$F_d = \frac{\rho_l R_v T}{D e_s(T)} \qquad F_k = \left(\frac{L}{R_v T} - 1\right)\frac{\rho_l L}{KT} \approx \frac{\rho_l L^2}{K R_v T^2}$$

K depends on *T, D depends on T and p*

$K \propto \mu$, μ *Dynamic viscosity of the air*

$D \propto {}^{\mu}/_{\rho}$, ${}^{\mu}/_{\rho} = \nu$ *Kinematic viscosity of the air*

$$\mu(T) = 1.72 \cdot 10^{-5} \frac{393}{T + 120}\left(\frac{T}{273}\right)^{3/2}$$

$$K = 4.1868 \cdot 10^{-3}[5.69 + 0.017(T - T_0)] \; Wm^{-1}K^{-1}$$

$$D = 2.11 \cdot 10^{-5}\left(\frac{T}{T_0}\right)^{1.94}\left(\frac{p_0}{p}\right) \; m^2 s^{-1}$$

$$T_0 = 273.15\,K; \quad p_0 = 1013.25\,hPa$$

For small solution droplets, curvature (Kelvin) and solute (Raoult) effects strongly modify the equilibrium surface supersaturation; consequently, the nonlinear growth equation has no closed-form solution and must be integrated numerically.

$$r\frac{dr}{dt} = \frac{(S-1) - \frac{A(T)}{r} + \frac{\kappa r_d^3}{r^3}}{[F_d + F_k]}$$

$$r\frac{dr}{dt} = \frac{S - \frac{r^3 - r_d^3}{r^3 - (1-\kappa)r_d^3}\exp\left(\frac{A(T)}{r}\right)}{[F_d + F_k]}$$

Numerical solution

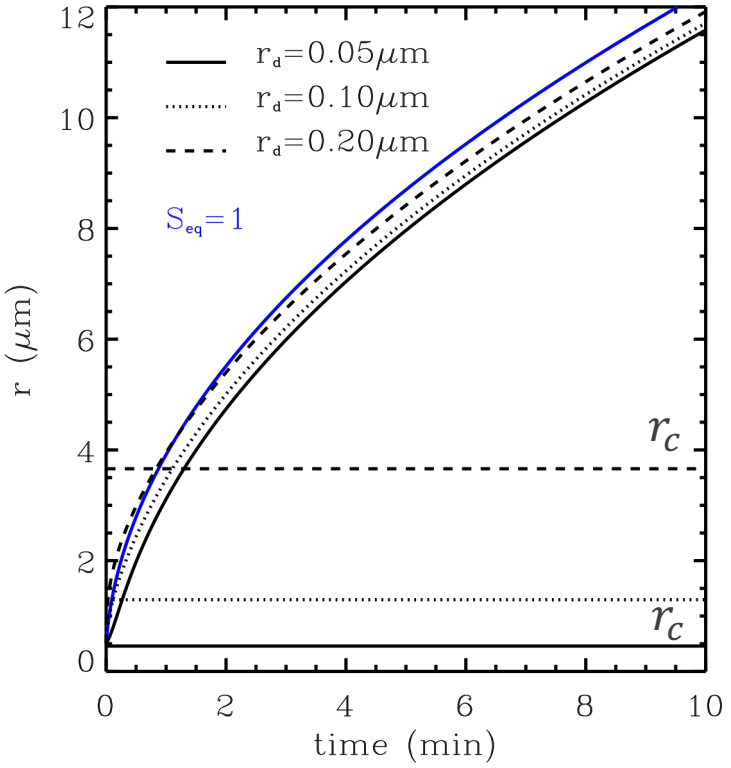
The figure shows the time necessary for a droplet to grow from the initial radius *0,5μm*, for different initial dry radii of the condensation nucleus ($\kappa = 0.67$).

Supersaturation 0,2%, p=900hPa, T=273 K

Droplets formed on big CCN grow initially faster (shorter time) than droplets formed an small CCN.

From a certain drop's diameter (~10 μm) the rate of growth is similar.

r_d (μm)	Time (s) to grow from $r = 0.5$ μm to 2 μm	Time (s) to grow from $r = 10$ μm to 12 μm
0.05	32.7	186.78
0.1	22.9	186.74
0.2	10.8	186.39



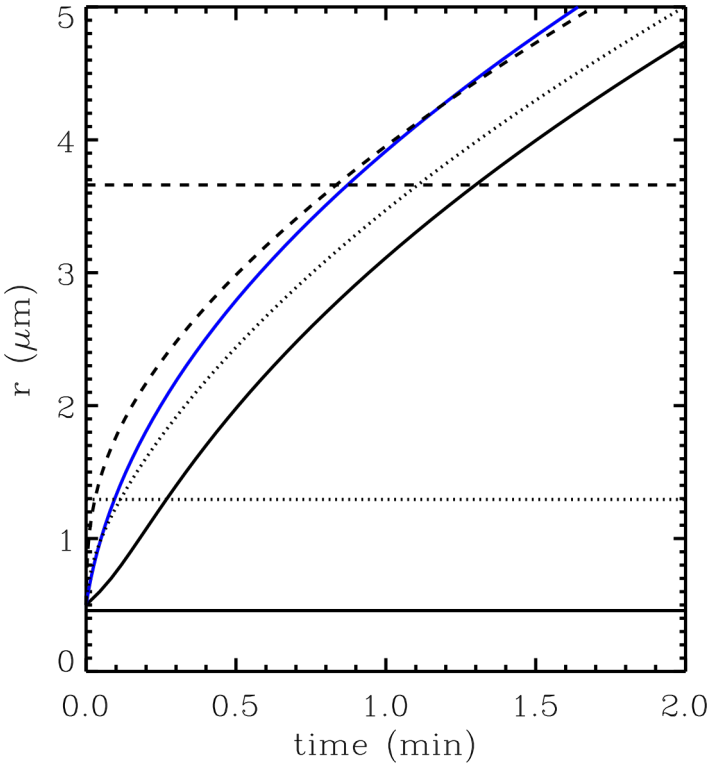
$$r\frac{dr}{dt} = \frac{(S-1) - \frac{A}{r} + \frac{\kappa r_d^3}{r^3}}{[F_d + F_k]}$$

$$r\frac{dr}{dt} = \frac{S - \frac{r^3 - r_d^3}{r^3 - (1-\kappa)r_d^3}\exp\left(\frac{A(T)}{r}\right)}{[F_d + F_k]}$$

Numerical solution

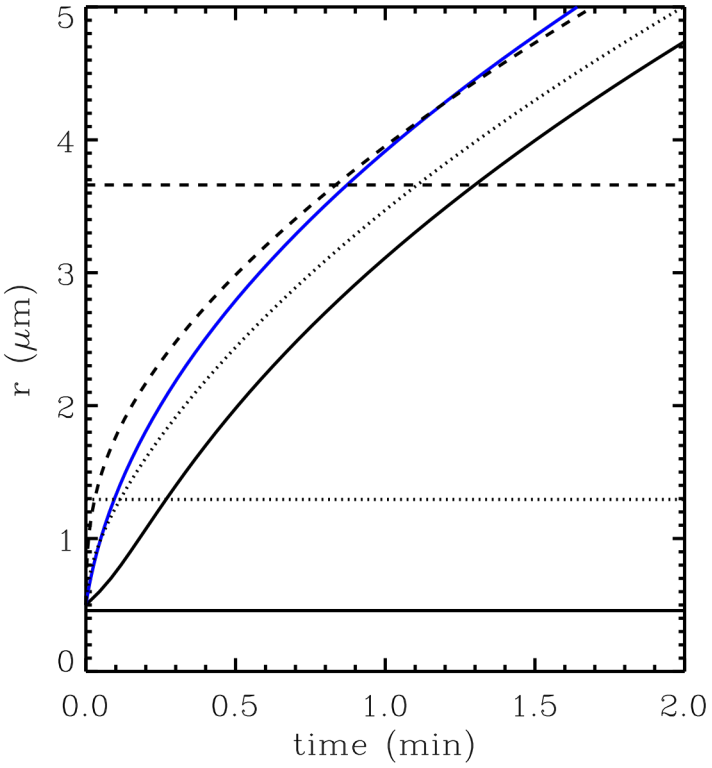
Droplets formed on big CCN grow initially faster (shorter time) than droplets formed an small CCN.

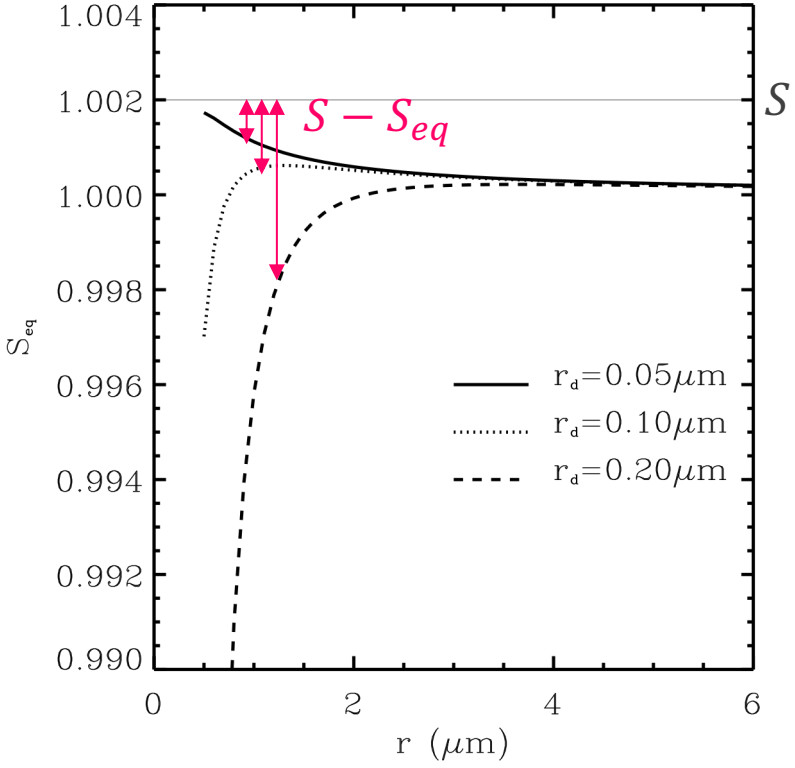
$r_d\ (\mu m)$	Time (s) to grow from $r = 0.5\ \mu m$ to $2\ \mu m$	Time (s) to grow from $r = 10\ \mu m$ to $12\ \mu m$
0.05	32.7	186.78
0.1	22.9	186.74
0.2	10.8	186.39



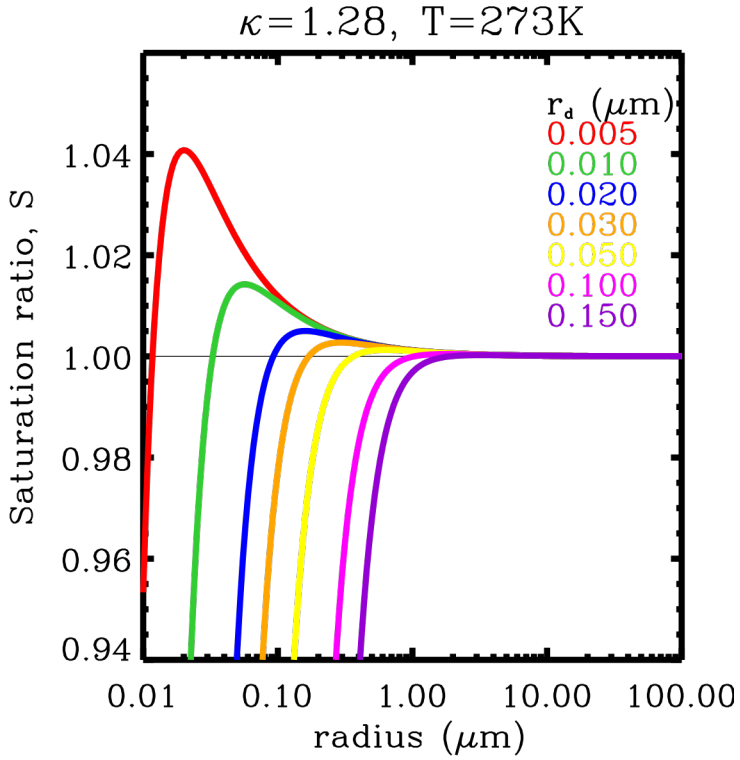
$$r\frac{dr}{dt} = \frac{(S - S_{eq})}{[F_d + F_k]}$$

$$S_{eq} = 1 + \frac{A}{r} - \frac{\kappa r_d^3}{r^3}$$





When a droplet becomes larger (approx. *10* μm in radius), the curvature and solution effects (A/r and $\kappa r_d^3/r^3$) become negligible compared to *S-1*, so $S_{eq} \approx 1$.



When a droplet becomes larger (approx. *10* μm in radius), the curvature and solution effects (A/r and $\kappa r_d^3/r^3$) become negligible compared to *S-1*, so $S_{eq} \approx 1$.

The equation for condensational growth can be solved analitically.

$$r\frac{dr}{dt} = \frac{S-1}{[F_d + F_k]}$$

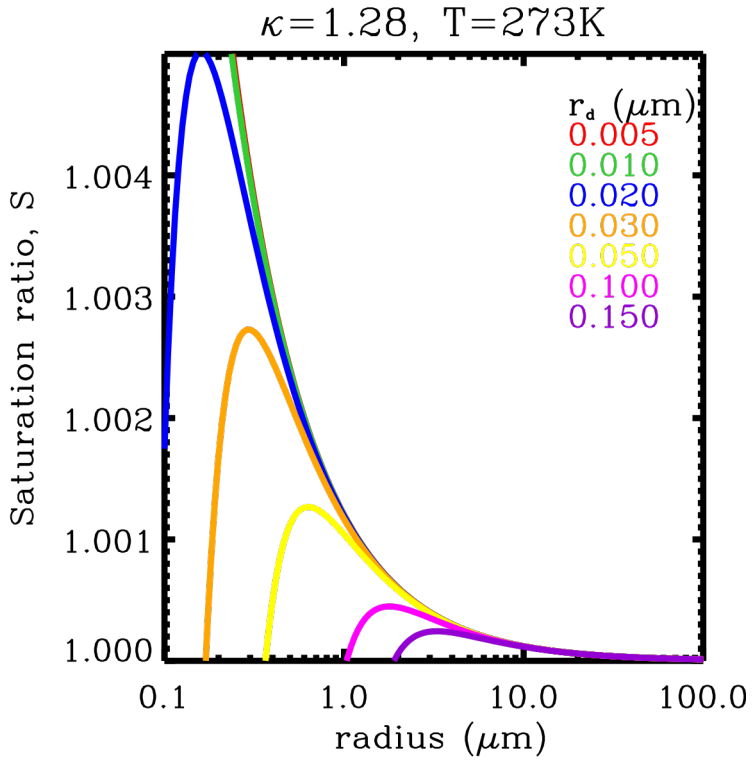
$$\frac{dr^2}{dt} = 2\xi(S-1)\,, \qquad \xi = \frac{1}{F_d + F_k}$$

The growth rate of a droplet surface area is independent of its radius.

$$r^2(t) - r^2(t_0) = 2\xi(S-1)\cdot(t-t_0)$$

$$r(t) = \sqrt{r^2(t_0) + 2\xi(S-1)\cdot(t-t_0)}$$

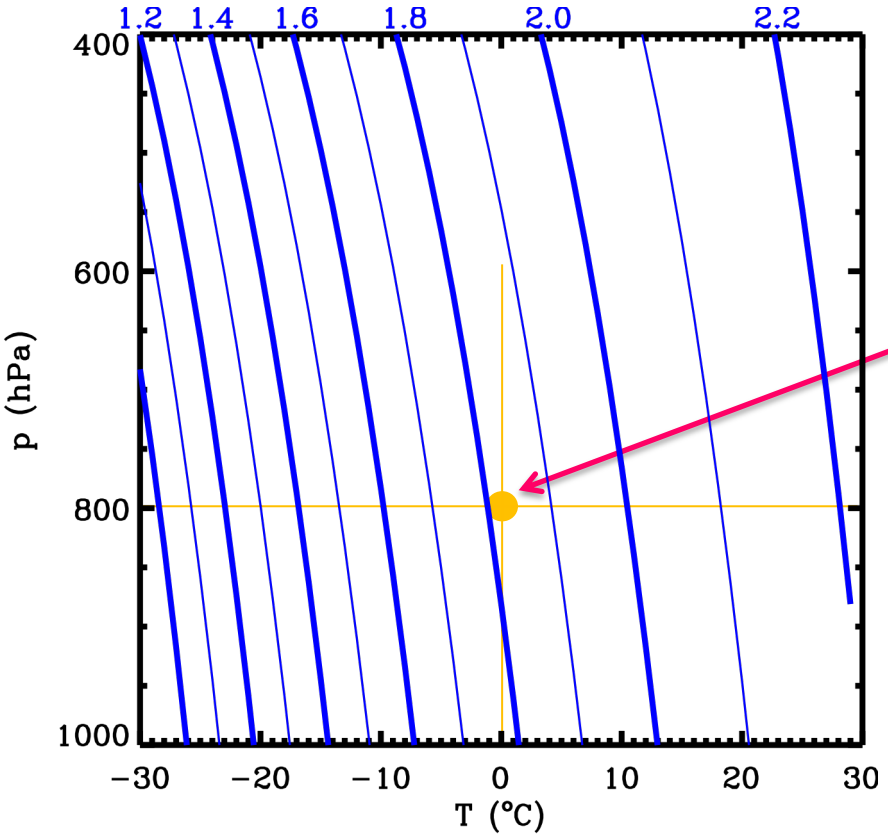
Smaller droplets grow faster in radius, causing the droplet size distribution to narrow during condensational growth.



$$\xi = \frac{1}{F_d + F_k}$$

ξ is a function of temperature and pressure

Solid lines represent $log_{10}\,\xi$



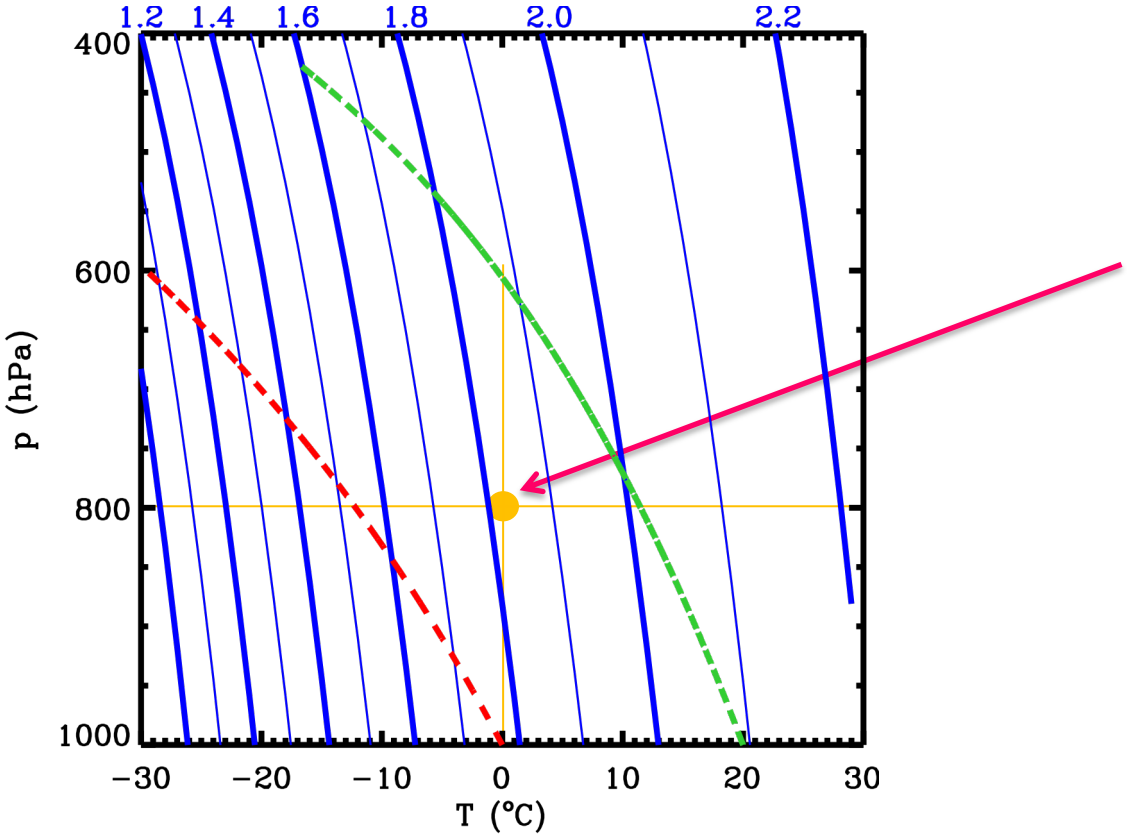
$$\xi = 10^{1.834} = 68.2\ \mu m^2 s^{-1}$$

$$\xi = \frac{1}{F_d + F_k}$$

ξ is a function of temperature and pressure

Solid lines represent $log_{10}\,\xi$

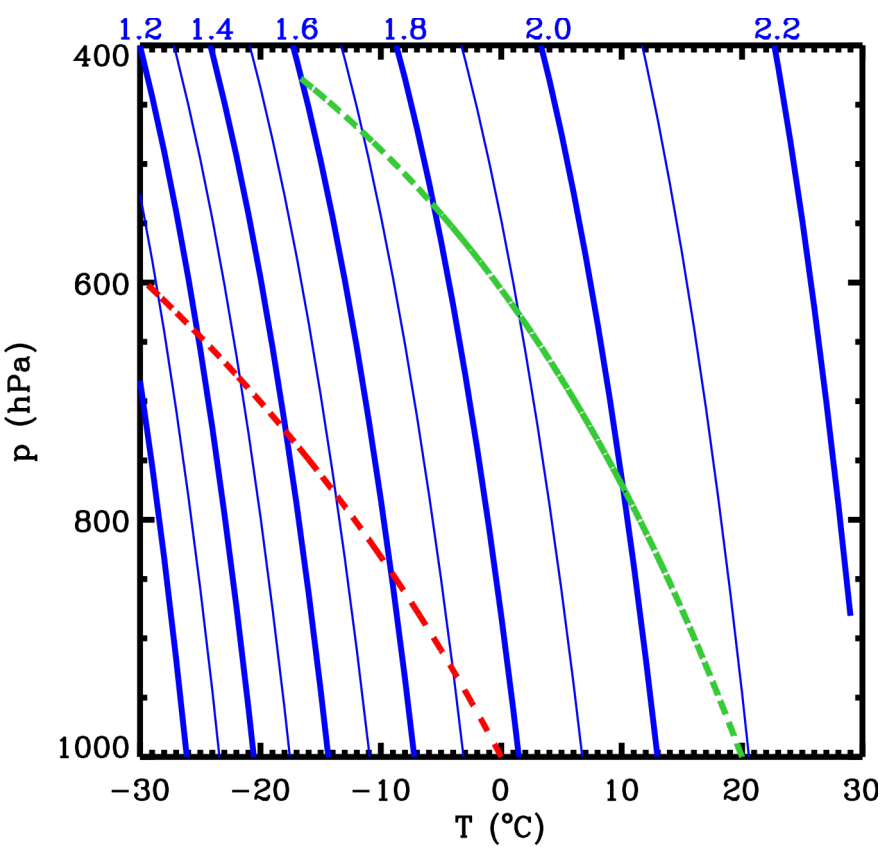
Dashed lines represent pseudoadiabats corresponding to *T= 0 ℃ and 20 ℃*

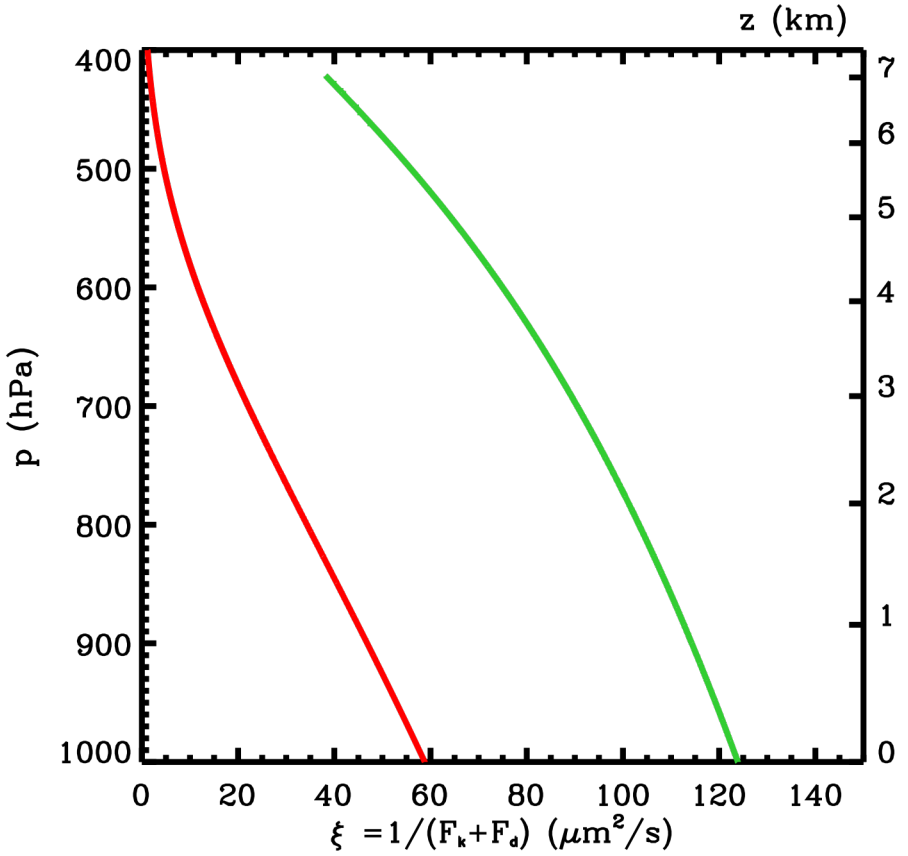


$p = 800\ hPa\,,\qquad T = 0℃$

$\xi = 10^{1.834} = 68.2\ \mu m^2 s^{-1}$

$$\xi = \frac{1}{F_d + F_k}$$

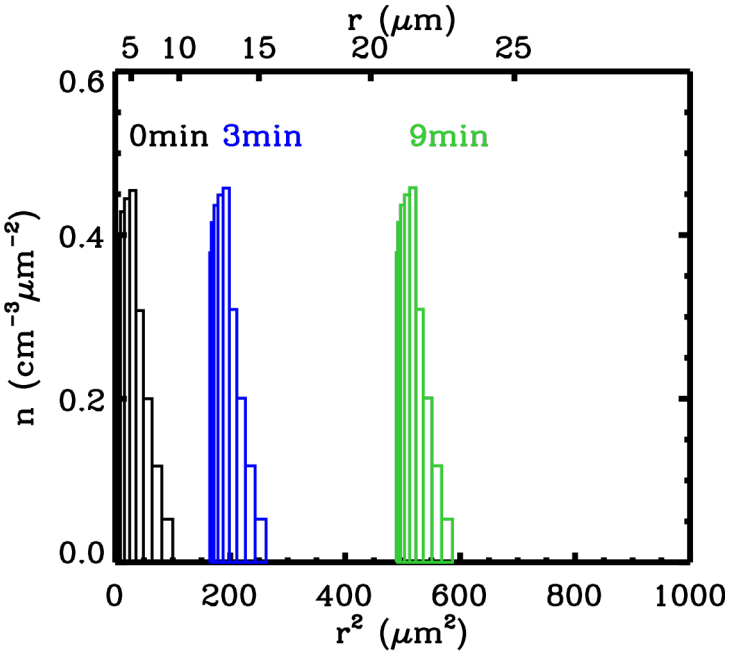




$$\frac{dr}{dt} = \xi \frac{S-1}{r}, \qquad \xi = \frac{1}{F_d + F_k}$$

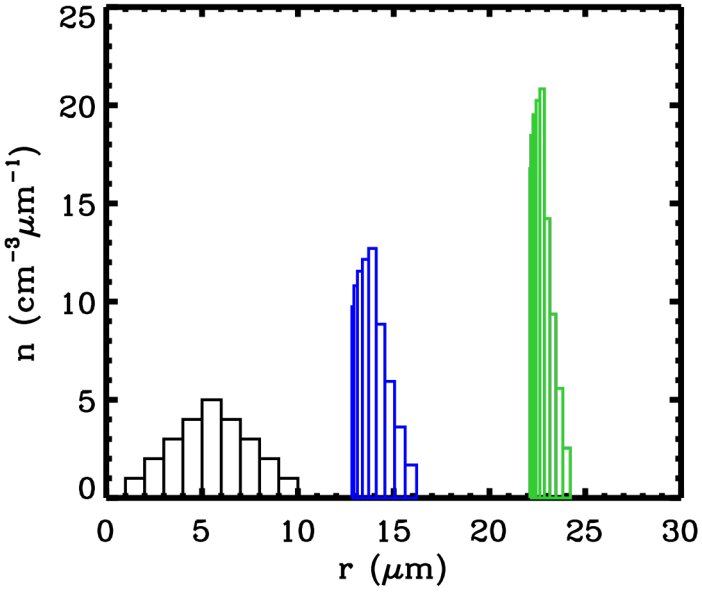
$$\rightarrow \quad \frac{dr^2}{dt} = 2\xi(S-1)$$

For fixed thermodynamic conditions, the growth rate of droplet surface area ($\propto r^2$) depends solely on ambient supersaturation .

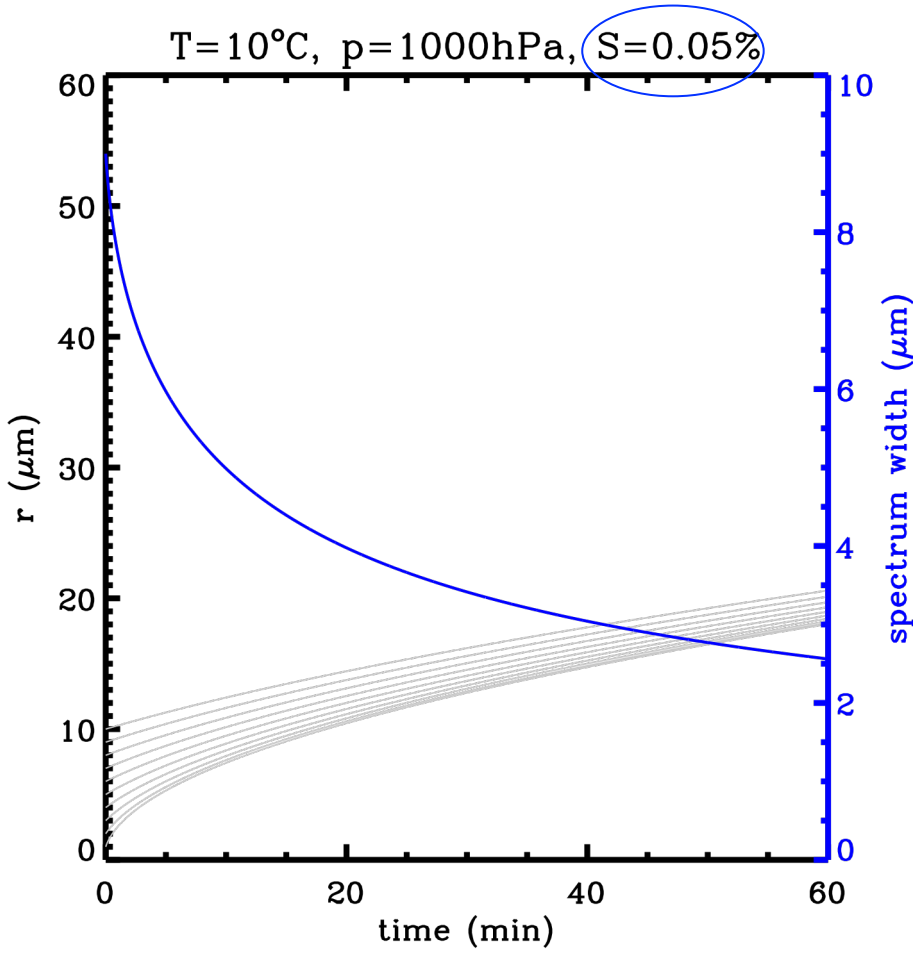


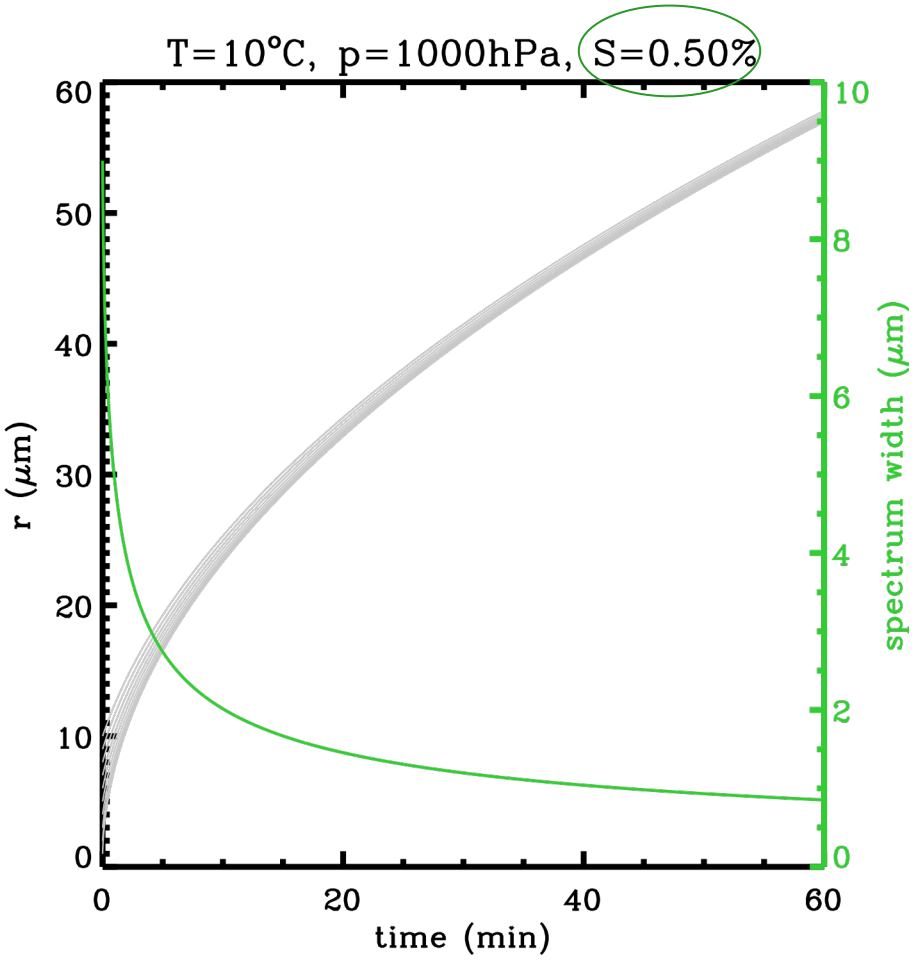
At the same supersaturation, smaller droplets grow faster in radius than larger ones.

Condensational growth leads to a narrowing of the droplet size distribution.

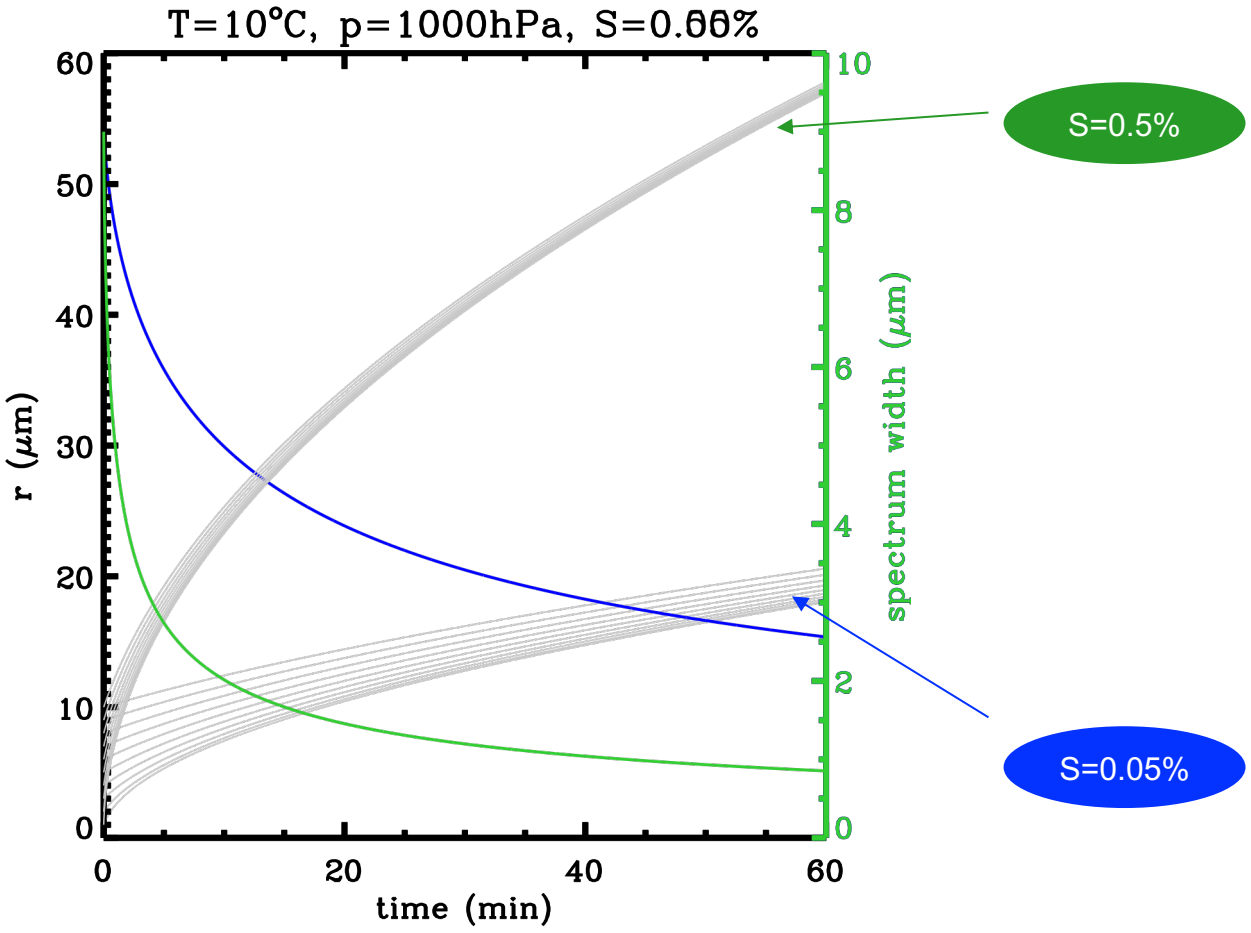


T=10°C, p=1000 hPa, S=0.5%





Under realistic cloud conditions, growth by water vapor diffusion seldom produces droplets with radii close to 20 μm due to the low ambient supersaturation and the limited time available for growth (~20 min).


T=10°C, p=1000hPa, S=0.66%
r (μm)
spectrum width (μm)
time (min)
0
20
40
60
0
10
20
30
40
50
60
0
2
4
6
8
10
S=0.5%
S=0.05%