Extra exercises for final exam

1. The (non-rotating) shallow water equations read

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g \nabla \eta,$$

$$\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h + h(\nabla \cdot \mathbf{u}) = 0,$$

where **u** is the horizontal velocity and h. Linearize these equations about a state of rest.

2. A westerly zonal barotropic flow at 43^{o}N is forced to rise over a north-south-oriented mountain barrier. Before striking the mountain, the westerly wind increases linearly toward the south at a rate of 10 m/s per 1000 km. The crest of the mountain range is 3 km high and the tropopause, located at 10 km, remains undisturbed.

   (a) What is the initial relative vorticity of the air?

   (b) What is its relative vorticity when it reaches the crest if it is deflected 5^o latitude toward the south during the forced ascent?

   (c) Supposing the current had uniform speed of 20 m/s while crossing crest, what would be the radius of curvature of the streamlines?

3. There is a counterclockwise rotating vortex in cyclostrophic balance, where the tangential velocity is given by the formula:

$$\bar{U}(R) = U_0 \left(\frac{R}{R_0 + 1} \right)^m \hat{e}_\theta.$$

U_0 and R_0 are initial velocity and initial distance from the center respectively. Calculate the circulation about a streamline at radius R, the vorticity at radius R, and the pressure at radius R. Assume here that P_0 is the pressure at R_0 and that the density is constant.

4. Consider the equations of motion in the form

$$\frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} = -\frac{1}{\rho} \nabla p + \bar{F}, \tag{1}$$

$$\nabla \cdot \bar{v} = -\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{2}$$

where **F** denotes friction forces (per unit mass).

(a) Use the vector identities

$$\bar{v} \cdot \nabla \bar{v} = \nabla \left(\frac{1}{2} \bar{v}^2 \right) + \bar{\omega} \times \bar{v},$$

and

$$\nabla \times (\bar{\omega} \times \bar{v}) = (\bar{v} \cdot \nabla)\bar{\omega} - (\bar{\omega} \cdot \nabla)\bar{v} + \bar{\omega}(\nabla \cdot \bar{v}) - \bar{v}(\nabla \cdot \bar{\omega}),$$

to show that:

$$\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} = (\bar{\omega} \cdot \nabla)\bar{v} - \bar{\omega}(\nabla \cdot \bar{v}) + \frac{1}{\rho^2}(\nabla \rho \times \nabla p) + \nabla \times \bar{F}.$$

(b) Simplify the equation above for an inviscid, incompressible and barotropic flow.

(c) Use mass conservation [Eq. (2)] to show that

$$\frac{\mathrm{D}\tilde{\bar{\omega}}}{\mathrm{D}t} = (\tilde{\bar{\omega}} \cdot \nabla)\bar{v} + \frac{1}{\rho^3}(\nabla \rho \times \nabla p) + \frac{1}{\rho}\nabla \times \bar{F},$$

where $\tilde{\bar{\omega}} \equiv \bar{\omega}/\rho$.