Dynamics of the Atmosphere and the Ocean

Lecture 9

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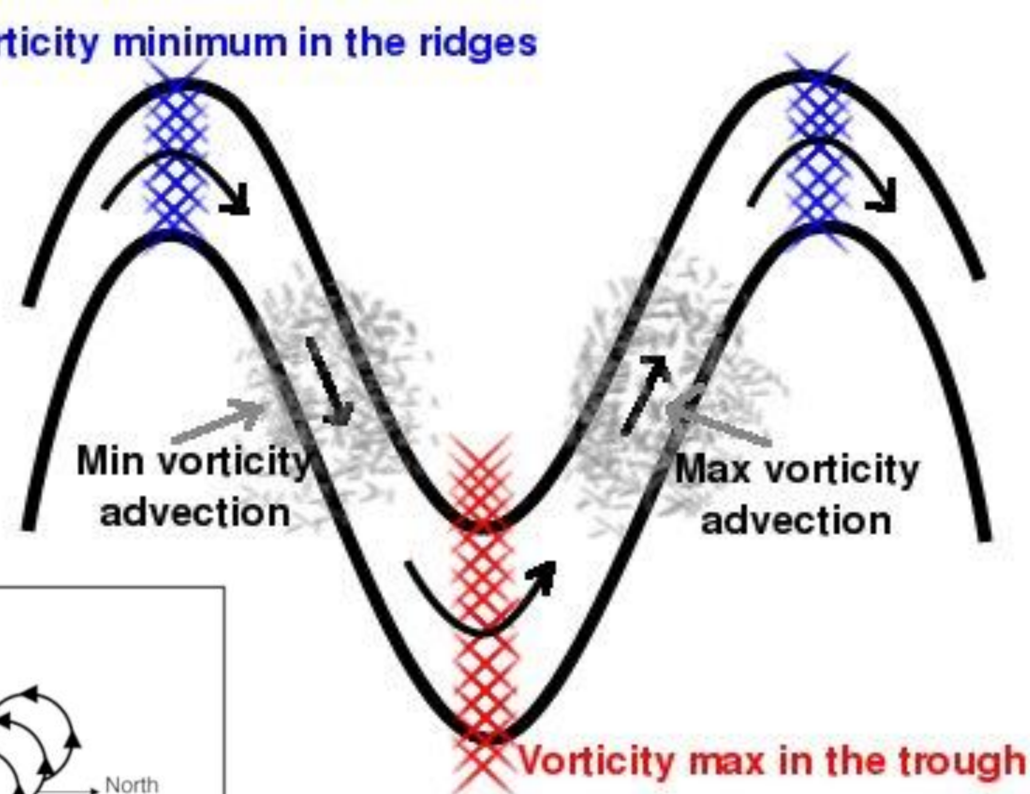
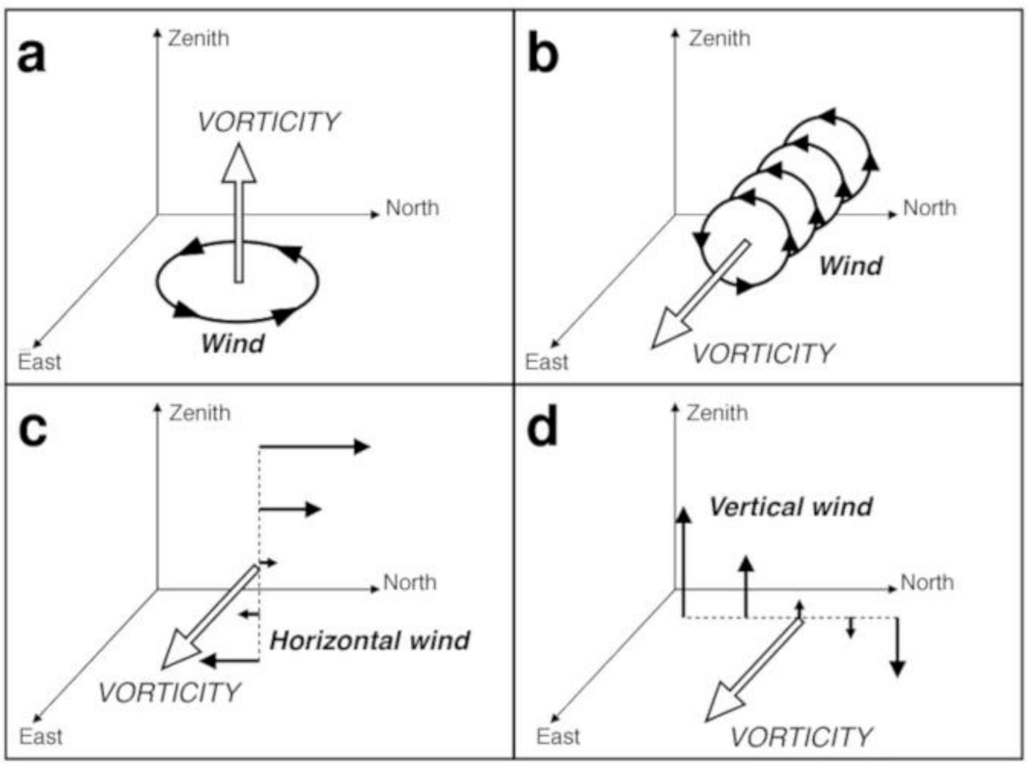
2025-2026 Fall



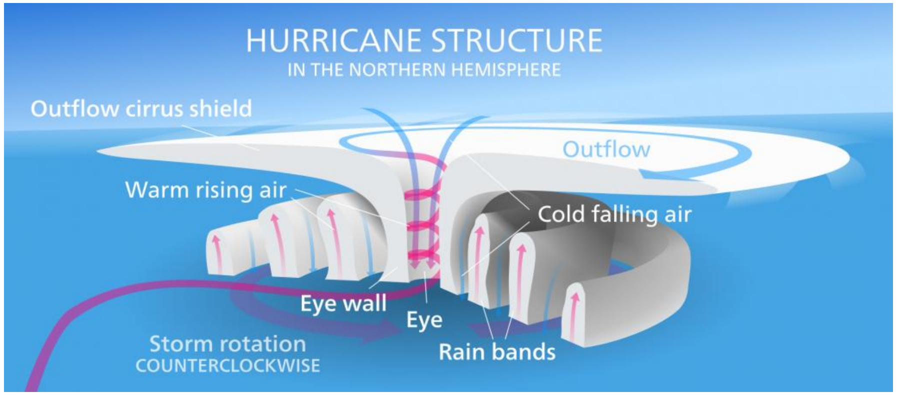


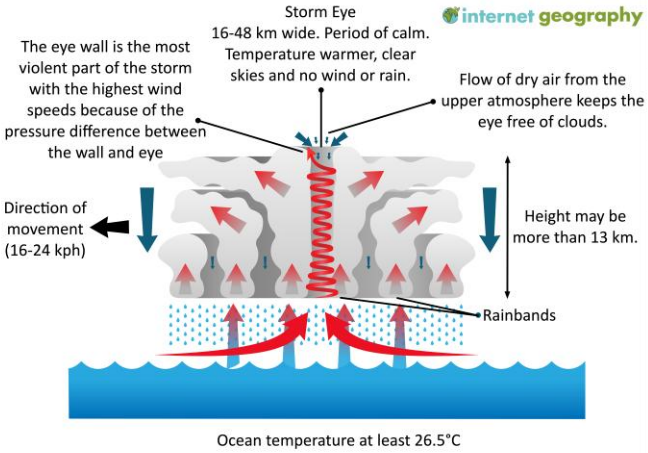
Recap

The vorticity equation:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

The above states that the rate of change of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence term, the tilting or twisting term, and the solenoidal term, respectively.





Recap

The vorticity equation:

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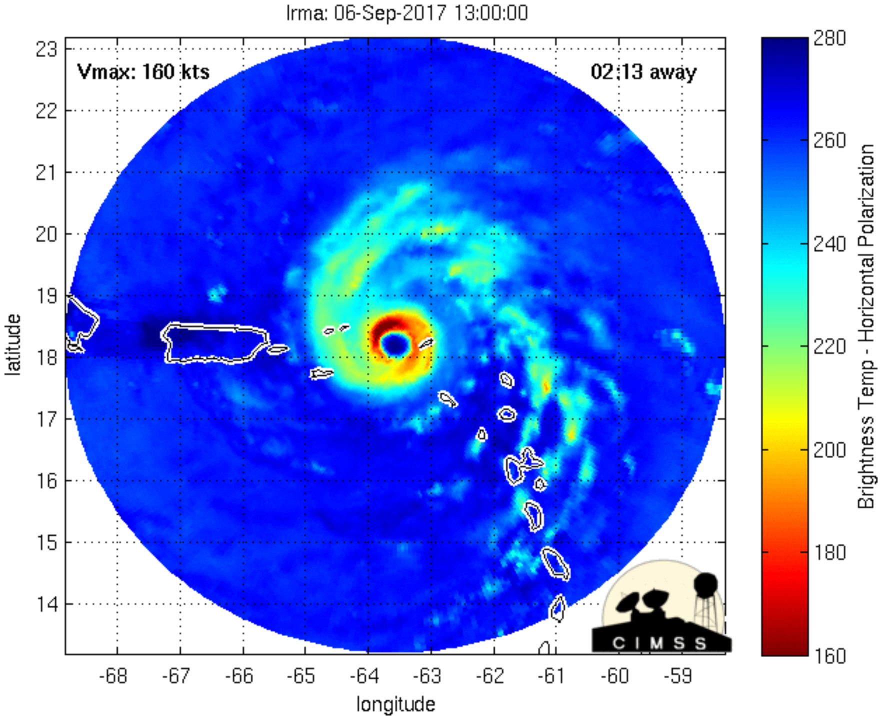


TC Irma, 2017

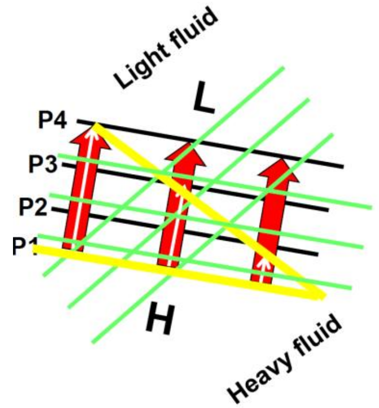
$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Solenoidal (baroclinic) term:

$$+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Another form: $\frac{1}{\rho^2}(\nabla\rho \times \nabla p) \cdot \boldsymbol{k}$

Vorticity generated due to intersection of density and pressure surfaces (baroclinic environment). Vorticity is generated when density and pressure gradients are not aligned. It causes stronger movement in a regions where gradients are larger and weaker movement in regions where gradients are (relatively) smaller, resulting in spin (vorticity).



Prof. Jin-Yi Yu

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \boxed{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}$$

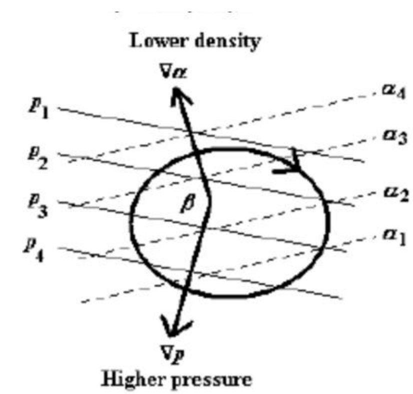
Solenoidal term:

$$+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

is just the microscopic equivalent of the solenoidal term in the circulation theorem.

To explain recall definition of **circulation** about a closed contour in a fluid as the line integral evaluated along the contour of the component of the velocity vector that is locally tangent to the contour:

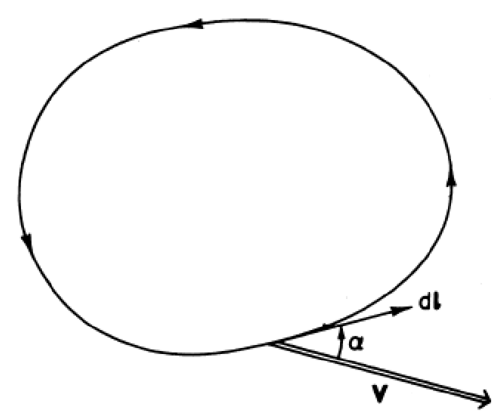
$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \oint |\mathbf{U}| \cos\alpha \, dl$$



where **l**(s) is a position vector extending from the origin to the point s(x, y, z) on the contour C, and d**l** represents the limit of

δl = l(s +δs)−l(s) as δs → 0.

By convention the circulation is taken to be positive if C > 0 for counterclockwise integration around the contour.



$$+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

That **circulation** is a measure of rotation is demonstrated readily by considering a **circular ring of fluid of radius R** in solid-body rotation at angular velocity about the z axis. In this case, U=Ω×R, where R is the distance from the axis of rotation to the ring of fluid. Thus the circulation about the ring is given by

$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \int_0^{2\pi} \Omega R^2 d\lambda = 2\Omega\pi R^2$$

In this case the circulation is just 2π times the angular momentum of the fluid ring.

Note also that $C/(\pi R^2)=2\Omega$ so that the circulation divided by the area enclosed by the loop is just twice the angular speed of rotation of the ring.
Unlike angular momentum or angular velocity, circulation can be computed without reference to an axis of rotation.

The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles. In the absolute coordinate system the result (neglecting viscous forces; aparent forces vanish) is

$$\frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r$$

$$\oint \frac{D_a \mathbf{U}_a}{Dt} \bullet d\mathbf{l} = -\oint \frac{\nabla p \bullet d\mathbf{l}}{\rho} - \oint \nabla\Phi \bullet d\mathbf{l}$$

$$+\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\oint \frac{D_a\mathbf{U}_a}{Dt}\bullet d\mathbf{l} = -\oint \frac{\boldsymbol{\nabla}_p \bullet d\mathbf{l}}{\rho} - \oint \boldsymbol{\nabla}\Phi \bullet d\mathbf{l}$$

In the above $-\nabla\Phi = \mathbf{g} = -g\mathbf{k}$.

The integrand on the **left-hand side** can be rewritten in the form:

$$\frac{D_a\mathbf{U}_a}{Dt}\bullet d\mathbf{l} = \frac{D}{Dt}(\mathbf{U}_a \bullet d\mathbf{l}) - \mathbf{U}_a \bullet \frac{D_a}{Dt}(d\mathbf{l})$$

$$D_a\mathbf{l}/Dt \equiv \mathbf{U}_a$$

$$\frac{D_a\mathbf{U}_a}{Dt}\bullet d\mathbf{l} = \frac{D}{Dt}(\mathbf{U}_a \bullet d\mathbf{l}) - \mathbf{U}_a \bullet d\mathbf{U}_a$$

Substituting to the equation on the top and using the fact that the line integral about a closed loop of a perfect differential is zero, so that

$$\oint \boldsymbol{\nabla}\Phi \bullet d\mathbf{l} = \oint d\Phi = 0$$

$$+\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\oint \frac{D_a \mathbf{U}_a}{Dt}\bullet d\mathbf{l} = -\oint \frac{\nabla p \bullet d\mathbf{l}}{\rho} - \oint \nabla\Phi \bullet d\mathbf{l}$$

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$$D_a\mathbf{l}/Dt \equiv \mathbf{U}_a$$

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Substituting to the equation on the top and using the fact that the line integral about a closed loop of a perfect differential is zero, so that

$$\oint \nabla\Phi \bullet d\mathbf{l} = \oint d\Phi = 0$$

One finally gets **the circulation theorem** in a form:

$$+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\frac{DC_a}{Dt}=\frac{D}{Dt}\oint \mathbf{U}_a \bullet d\mathbf{l} = -\oint \rho^{-1} dp$$

The above was obtained using $\oint \mathbf{U}_a \bullet d\mathbf{U}_a = \frac{1}{2}\oint d(\mathbf{U}_a \bullet \mathbf{U}_a) = 0$

In the circulation theorem the rightmost term is the solenoidal one.
We may now apply Stokes' theorem to the solenoidal term to get:

$$-\oint \alpha dp \equiv -\oint \alpha \boldsymbol{\nabla} p \bullet d\mathbf{l} = -\iint_A \boldsymbol{\nabla} \times (\alpha \boldsymbol{\nabla} p) \bullet \mathbf{k} dA$$

where A is the horizontal area bounded by the curve **l**.
Applying the vector identity $\nabla \times (\alpha\nabla p) \equiv \nabla\alpha \times \nabla p$, the equation becomes:

$$-\oint \alpha dp = -\iint_A (\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla} p) \bullet \mathbf{k} dA$$

However, since $\alpha=1/\rho$ the solenoidal term in the vorticity equation can be written

$$-\left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right) = -(\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla} p) \bullet \mathbf{k}$$

and the **solenoidal term in the vorticity equation is just the limit of the solenoidal term in the circulation theorem divided by the area when the area goes to zero**.

Discussion of circulation theorem.

Consider the circulation theorem in the form:

$$\frac{DC_a}{Dt} = \frac{D}{Dt}\oint \mathbf{U}_a \bullet d\mathbf{l} = -\oint \rho^{-1} dp$$

For meteorological analysis, it is more convenient to work with the relative circulation C rather than the absolute circulation, as a portion of the absolute circulation, C_e , is due to the rotation of the earth about its axis.

To compute C_e , we apply Stokes' theorem to the vector $\mathbf{U_e}$, where $\mathbf{U_e} = \mathbf{\Omega} \times \mathbf{r}$ is the velocity of the earth at the position r:

$$C_e = \oint \mathbf{U}_e \bullet d\mathbf{l} = \int_A\int (\nabla \times \mathbf{U}_e) \bullet \mathbf{n} dA$$

Here **n** is normal to the area A.

Using vector identity:

$$\nabla \times \mathbf{U}_e = \nabla \times (\mathbf{\Omega} \times \mathbf{r}) = \nabla \times (\mathbf{\Omega} \times \mathbf{R}) = \mathbf{\Omega} \nabla \bullet \mathbf{R} = 2\mathbf{\Omega}$$

one obtains:

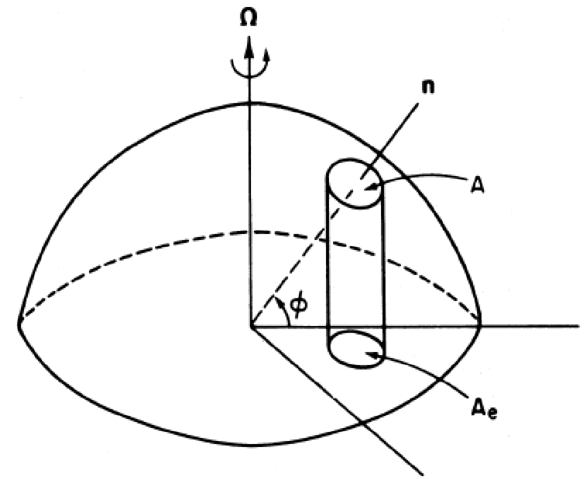
$$(\nabla \times \mathbf{U}_e) \bullet \mathbf{n} = 2\Omega \sin\phi \equiv f$$

Hence, the circulation in the horizontal plane due to the rotation of the earth is

$C_e = 2\Omega \langle \sin \varphi \rangle A = 2\Omega A_e$

where $\langle \sin \varphi \rangle$ denotes an average over the area element A and A_e is the projection of A in the equatorial plane as illustrated.
Thus, the relative circulation may be expressed as

$C = C_a - C_e = Ca - 2A\Omega_e$



Differentiating following the motion (D/Dt) we obtain the Bjerknes circulation theorem:

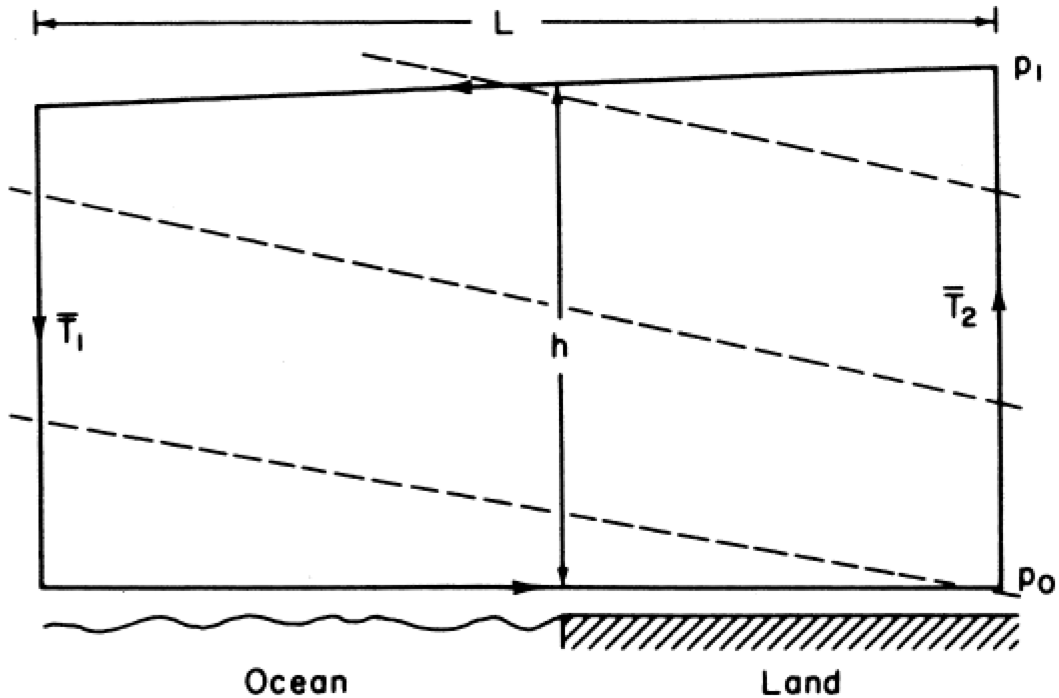
$$\oint \frac{D_a \mathbf{U}_a}{Dt} \bullet d\mathbf{l} = -\oint \frac{\nabla p \bullet d\mathbf{l}}{\rho} - \oint \nabla \Phi \bullet d\mathbf{l}$$

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt}$$

For a barotropic fluid, the above can be integrated following the motion from an initial state (designated by subscript 1) to a final state (designated by subscript 2), yielding the circulation change:

$C_2 - C_1 = -2\Omega(A_2 \langle \sin \varphi_2 \rangle - A_1 \langle \sin \varphi_1 \rangle)$

In a baroclinic fluid, circulation may be generated by the pressure-density solenoid term. This process can be illustrated effectively by considering the development of a sea breeze circulation, as shown:



Substituting the ideal gas law into circulation theorem we obtain

$$\frac{DC_a}{Dt} = -\oint RT d \ln p$$

$$\frac{DC_a}{Dt} = R \ln\left(\frac{p_0}{p_1}\right)(\overline{T}_2 - \overline{T}_1) > 0$$

Vorticity equation – practical application

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Quarterly Journal of the Royal Meteorological Society RMetS
RESEARCH ARTICLE

Dynamical propagation and growth mechanisms for convectively coupled equatorial Kelvin waves over the Indian Ocean

Adrian J. Matthews

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Quarterly Journal of the Royal Meteorological Society RMetS
RESEARCH ARTICLE

A vorticity budget for theoretical and convectively coupled equatorial Rossby waves: Dynamical propagation and growth mechanisms

Adrian J. Matthews

Vorticity equation can be used to study propagation and growth/decay mechanisms of real atmospheric features.

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

2.1 | Vorticity budget

The vorticity equation for flow on a quasi-horizontal pressure level in Cartesian coordinates can be written as

$$\frac{\partial \zeta}{\partial t} = \underbrace{-u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - \omega\frac{\partial \zeta}{\partial p}}_{\text{advection of relative vorticity}} \quad \underbrace{-\zeta D - fD}_{\text{vortex stretching}} \quad \underbrace{-\beta v}_{\text{advection of planetary vorticity}} \quad \underbrace{-\left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right)}_{\text{tilting/twisting}},$$

Applicable to indealized (analytical) models, modeling data (weather forecasts), observations, reanalyses…
Process-level diagnostics in a dynamical framework.

Coming back to vorticity....

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times \mathbf{V} = -\nabla_p \Phi$$

Vorticity equation in isobaric coordinates can be derived in vector form by operating on the momentum equation with the vector operator
k · ∇×, where ∇ now indicates the horizontal gradient on a surface of constant pressure.
and use the vector identity

$$(\mathbf{V} \bullet \nabla)\mathbf{V} = \nabla\left(\frac{\mathbf{V} \bullet \mathbf{V}}{2}\right) + \zeta\mathbf{k} \times \mathbf{V}$$

$\zeta = k \cdot (\nabla \times V)$

After these operations one obtains:

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla\left(\frac{\mathbf{V} \bullet \mathbf{V}}{2} + \Phi\right) - (\zeta + f)\mathbf{k} \times \mathbf{V} - \omega\frac{\partial \mathbf{V}}{\partial p}$$

We now apply the operator k · ∇× to the above..

Using the facts that for any scalar A, $\nabla \times \nabla A = 0$ and for any vectors a, b,

$\nabla \times (a \times b) = (\nabla \cdot b)\, a - (a \cdot \nabla)\, b - (\nabla \cdot a)\, b + (b \cdot \nabla)\, a$

we can eliminate the first term on the right and simplify the second term so that the ı

$$\frac{\partial \zeta}{\partial t} = -\mathbf{V} \bullet \nabla(\zeta + f) - \omega\frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla \bullet \mathbf{V} + \mathbf{k} \bullet \left(\frac{\partial \mathbf{V}}{\partial p} \times \nabla\omega\right)$$

There is no solenoidal term in pressure coordinates!

Scale analysis of the vorticity equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

We simplify the equations of motion for synoptic-scale motions by evaluating the order of magnitude of various terms.

The same technique can also be applied to the vorticity equation. Characteristic scales for the field variables (in the atmosphere!!!) are as follows:

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical scale
$L \sim 10^6\,\mathrm{m}$	length scale
$H \sim 10^4\,\mathrm{m}$	depth scale
$\delta p \sim 10\,\mathrm{hPa}$	horizontal pressure scale
$\rho \sim 1\,\mathrm{kg\,m^{-3}}$	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
$L/U \sim 10^5\,\mathrm{s}$	time scale
$f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$	Coriolis parameter
$\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$	"beta" parameter

Using these scales to evaluate the magnitude of the terms in vorticity equation, we note that

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \lesssim \frac{U}{L} \sim 10^{-5}\mathrm{s}^{-1}$$

$$\zeta/f_0 \lesssim U/(f_0 L) \equiv \mathrm{Ro} \sim 10^{-1}$$

For **midlatitude** synoptic-scale systems, the relative vorticity is often small (order Rossby number) compared to the planetary vorticity and ζ may be neglected compared to f in the divergence term in the vorticity equation:

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

This approximation does not apply near the center of intense cyclonic storms. In such systems $|\zeta / f| \sim 1$, and the relative vorticity should be retained. Where else?

The magnitudes of the various terms in the vorticity equation can now be estimated as:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial \zeta}{\partial t}, u\frac{\partial \zeta}{\partial x}, v\frac{\partial \zeta}{\partial y} \sim \frac{U^2}{L^2} \sim 10^{-10}\ \mathrm{s}^{-2}$$

$$w\frac{\partial \zeta}{\partial z} \sim \frac{WU}{HL} \sim 10^{-11}\ \mathrm{s}^{-2}$$

$$v\frac{df}{dy} \sim U\beta \sim 10^{-10}\ \mathrm{s}^{-2}$$

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim \frac{f_0 U}{L} \sim 10^{-9}\ \mathrm{s}^{-2}$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \lesssim \frac{WU}{HL} \sim 10^{-11}\ \mathrm{s}^{-2}$$

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \lesssim \frac{\delta\rho\delta p}{\rho^2 L^2} \sim 10^{-11}\ \mathrm{s}^{-2}$$

The inequality is used in the last three terms because the two parts of the expression might partially cancel and the actual magnitude would be less than indicated.
In fact, this must be the case for the divergence term because if ∂u/∂x and ∂v/∂y were not nearly equal and opposite, the divergence term would be an order of magnitude greater than any other term and the equation could not be satisfied.

Scale analysis of the vorticity equation indicates that synoptic-scale motions must be quasi-nondivergent.

The divergence term will be small enough to be balanced by the vorticity advection terms only if

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right| \lesssim 10^{-6}\ \mathrm{s}^{-1}$$

so that the horizontal divergence must be small compared to the vorticity in synoptic-scale systems. From this and the definition of the Rossby number, we see that

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ f_0\right| \lesssim \mathrm{Ro}^2$$

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ \zeta\right| \lesssim \mathrm{Ro}$$

The ratio of the horizontal divergence to the relative vorticity is the same magnitude as the ratio of relative vorticity to planetary vorticity.

Retaining only the terms of order 10^{-10} s^{-2} in the **vorticity equation** yields the **approximate form valid for synoptic-scale motions**:

$$\frac{D_h(\zeta + f)}{Dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad \frac{D_h}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

The above equation states that the change of absolute vorticity following the horizontal motion on the synoptic scale is given approximately by the concentration/dilution of planetary vorticity caused by the convergence/divergence of the horizontal flow.

It is not accurate in intense cyclonic storms. For these the relative vorticity should be retained in the divergence term:

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

In the above the concentration or dilution of absolute vorticity that leads to changes in absolute vorticity following the motion.

The approximate forms above do not remain valid in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only ~100 km, and the vertical velocity scale is ~10 cm s^{-1} .
For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

VORTICITY IN BAROTROPIC FLUIDS

For a homogeneous incompressible fluid, the continuity equation simplifies because $\nabla \cdot U = 0$ or, in Cartesian coordinates to:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial w}{\partial z}$$

so that the vorticity equation may be written as:

$$\frac{D_h(\zeta + f)}{Dt} = (\zeta + f)\left(\frac{\partial w}{\partial z}\right)$$

We know that in a barotropic fluid the geostrophic wind is independent of height. Letting the vorticity be approximated by the geostrophic vorticity ζg and the wind by the geostrophic wind (ug , vg), we can integrate vertically from $z1$ to $z2$ to get

$$h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)\left[w(z_2) - w(z_1)\right]$$

Knowing that

$$w \equiv Dz/Dt \text{ and } h \equiv h(x, y, t),$$

$$w(z_2) - w(z_1) = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{D_h h}{Dt}$$

we get

$$\frac{D_h \ln(\zeta_g + f)}{Dt} = \frac{D_h \ln h}{Dt}$$

$$\frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0$$

The last one is the **barortopic potential vorticity equation**, derived in the other way in the previous lecture.

The Barotropic Vorticity Equation

If the flow is purely horizontal (w = 0), as is the case for barotropic flow in a fluid of constant depth, the divergence term vanishes and we obtain the barotropic vorticity equation:

$$\frac{D_h\left(\zeta_g + f\right)}{Dt} = 0$$

Indicating that absolute vorticity is conserved following the horizontal motion.

More generally, absolute vorticity is conserved for any fluid layer in which the divergence of the horizontal wind vanishes, without the requirement that the flow be geostrophic.

For horizontal motion that is nondivergent ($\partial u/\partial x + \partial v/\partial y = 0$), the flow field can be represented by a streamfunction ψ (x, y) defined so that the velocity components are given as $u = -\partial\psi/\partial y$, $v = +\partial\psi/\partial x$. The vorticity is then given by

$$\zeta = \partial v/\partial x - \partial u/\partial y = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \equiv \nabla^2\psi$$

The velocity field and the vorticity can both be represented in terms of the variation of the single scalar field ψ (x, y), and barotropic vorticity equation can be written as a prognostic equation for vorticity in the form:

$$\mathbf{V}_\psi \equiv \mathbf{k} \times \nabla\psi$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\mathbf{V}_\psi \bullet \nabla\left(\nabla^2\psi + f\right)$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -\mathbf{V}_\psi \bullet \nabla\left(\nabla^2\psi + f\right)$$

In the above Vψ ≡ k × ∇ψ is a nondivergent horizontal wind. The equation states that the local tendency of relative vorticity is given by the advection of absolute vorticity.

This equation can be solved numerically to predict the evolution of the streamfunction, and hence of the vorticity and wind field. In fact this was the firs numerical weather forecast!!
You can get the code of this forecast from: http://mathsci.ucd.ie/~plynch/eniac/ and run it on your cell phone.

Considering that **the flow in the mid-troposphere is often nearly nondivergent** on the synoptic scale, the above equation provides a surprisingly good model for short-term forecasts of the synoptic-scale 500-hPa flow field.