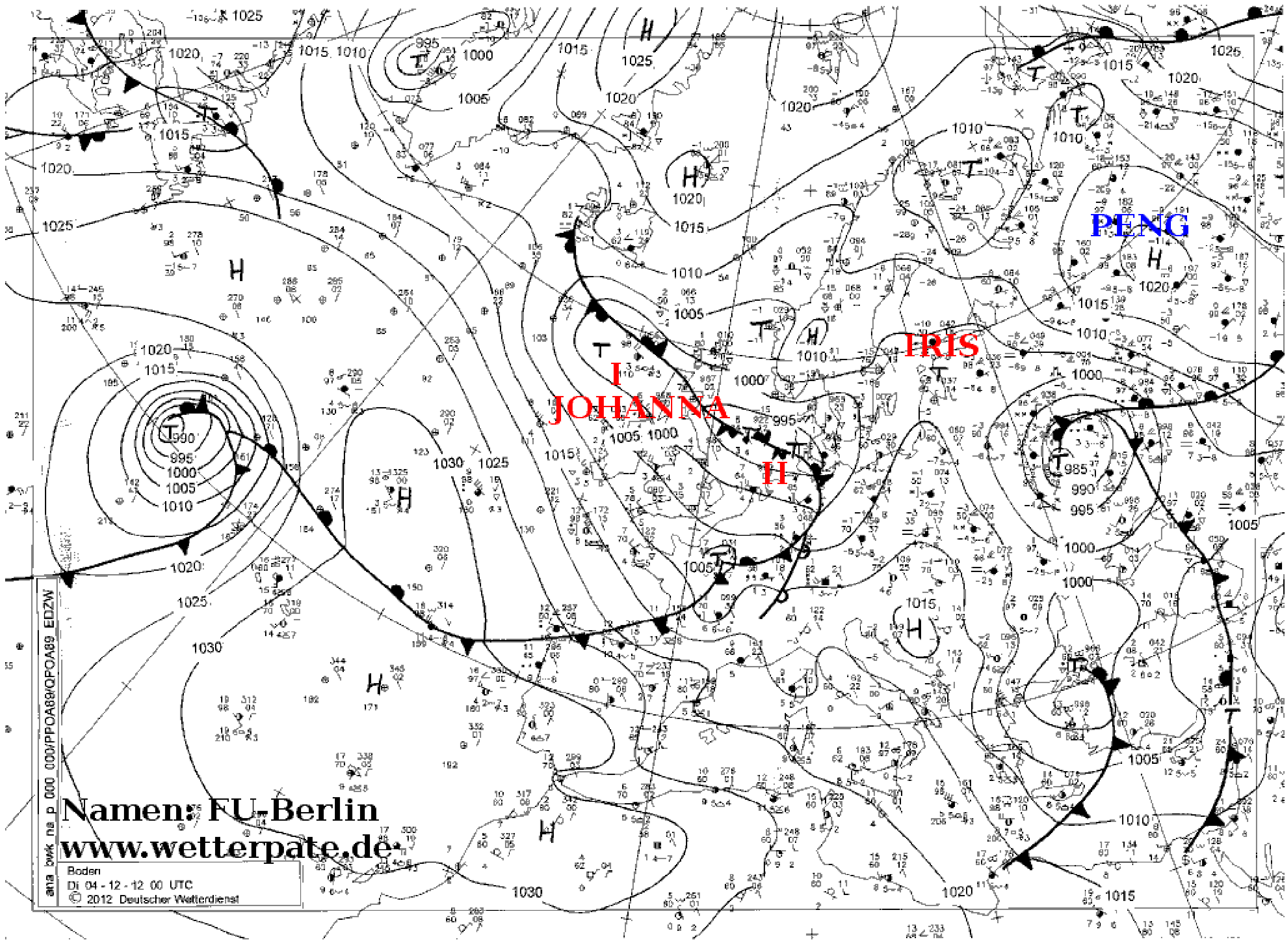
Dynamics of the Atmosphere and the Ocean

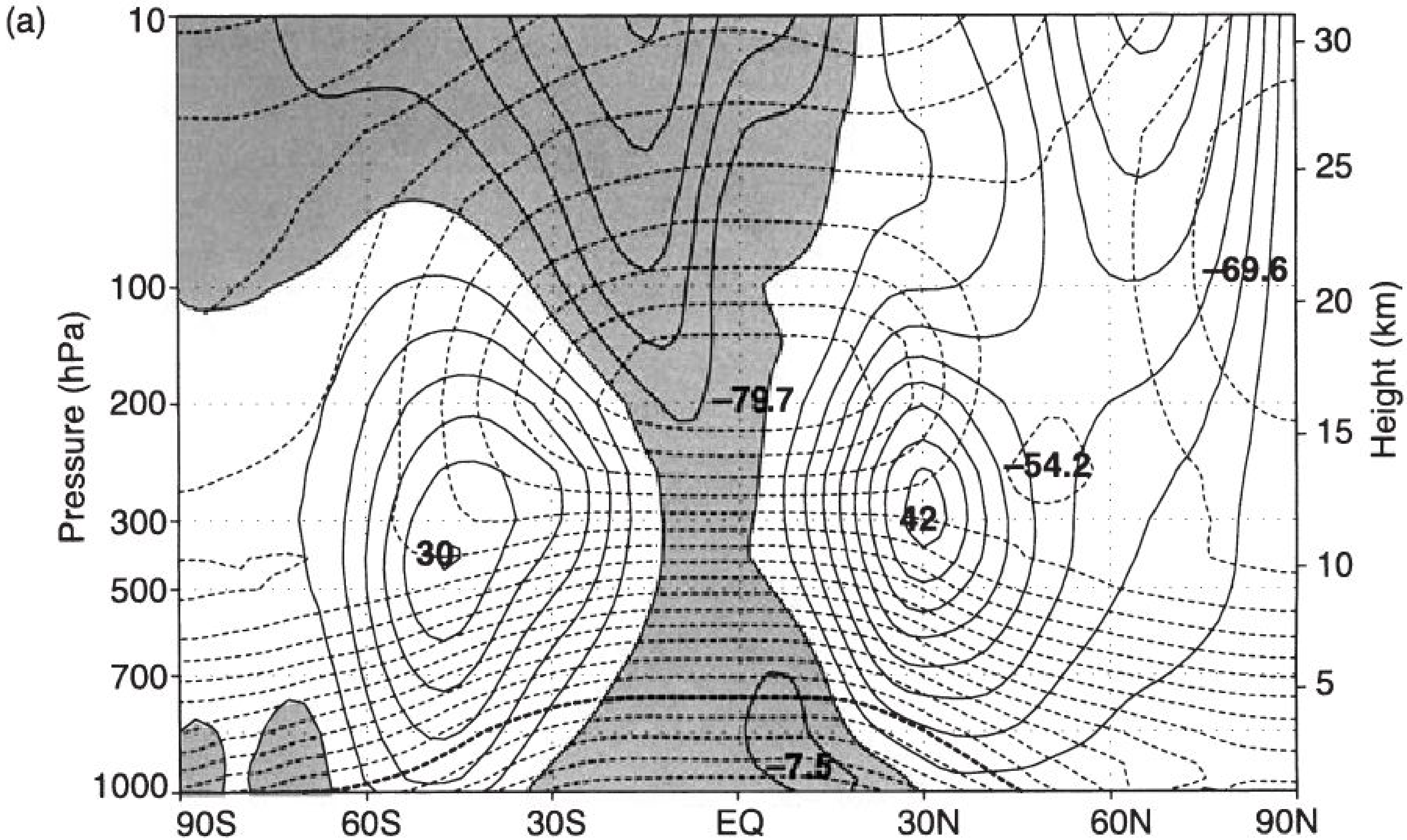
Lecture 9

Szymon Malinowski

2021 Fall

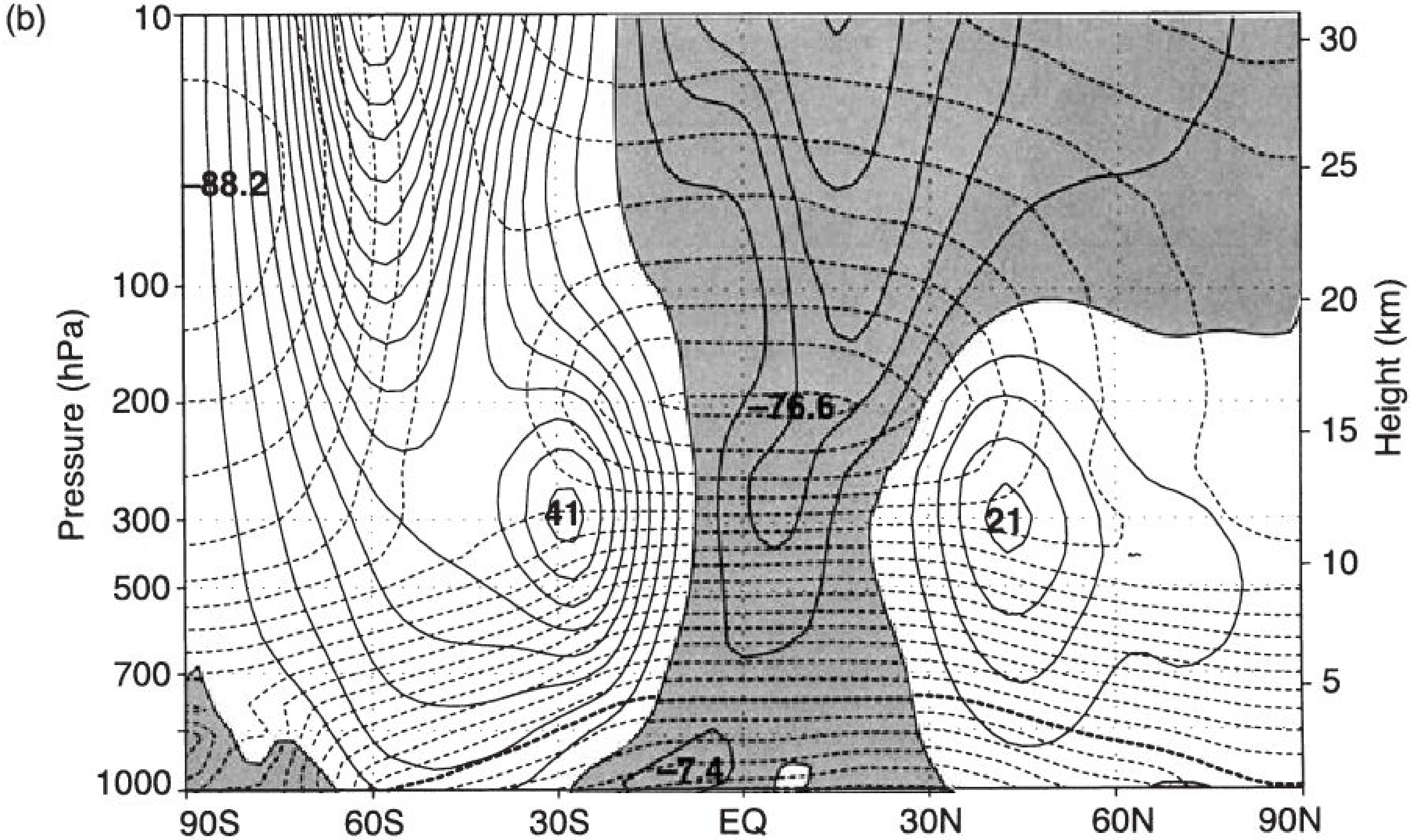


Synoptic-Scale Motions: Quasi-Geostrophic Analysis



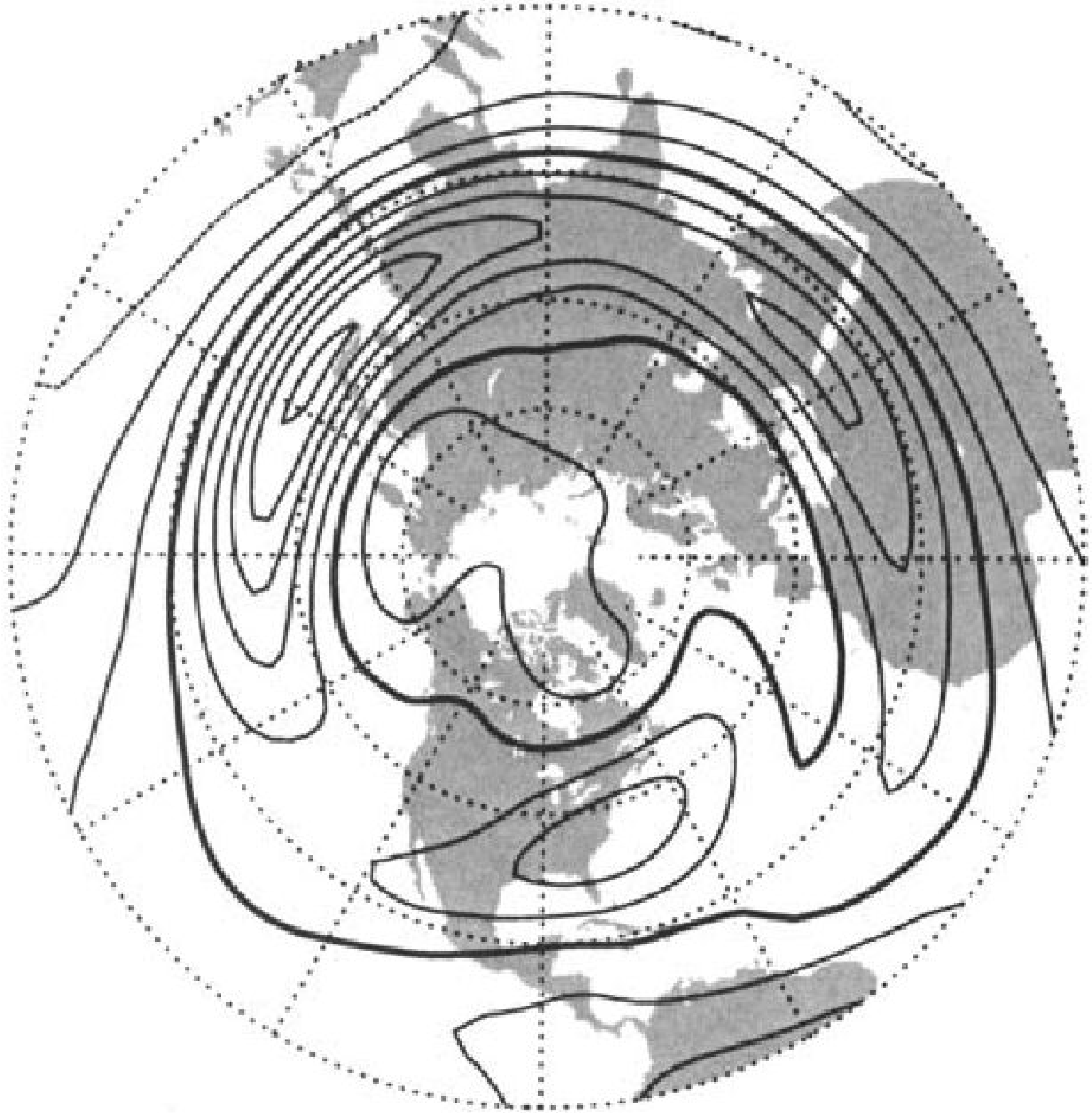
Meridional cross section of longitudinally and time-averaged zonal wind (solid contours, interval of ms^{-1}) and temperature (dashed contours, interval of 5 K) for December–February. Easterly winds are shaded and 0˚ C isotherm is darkened. Wind maxima shown in ms^{-1} , temperature minima shown in ˚C. (NCEP/NCAR reanalyses; after Wallace, 2003.)

Synoptic-Scale Motions: Quasi-Geostrophic Analysis



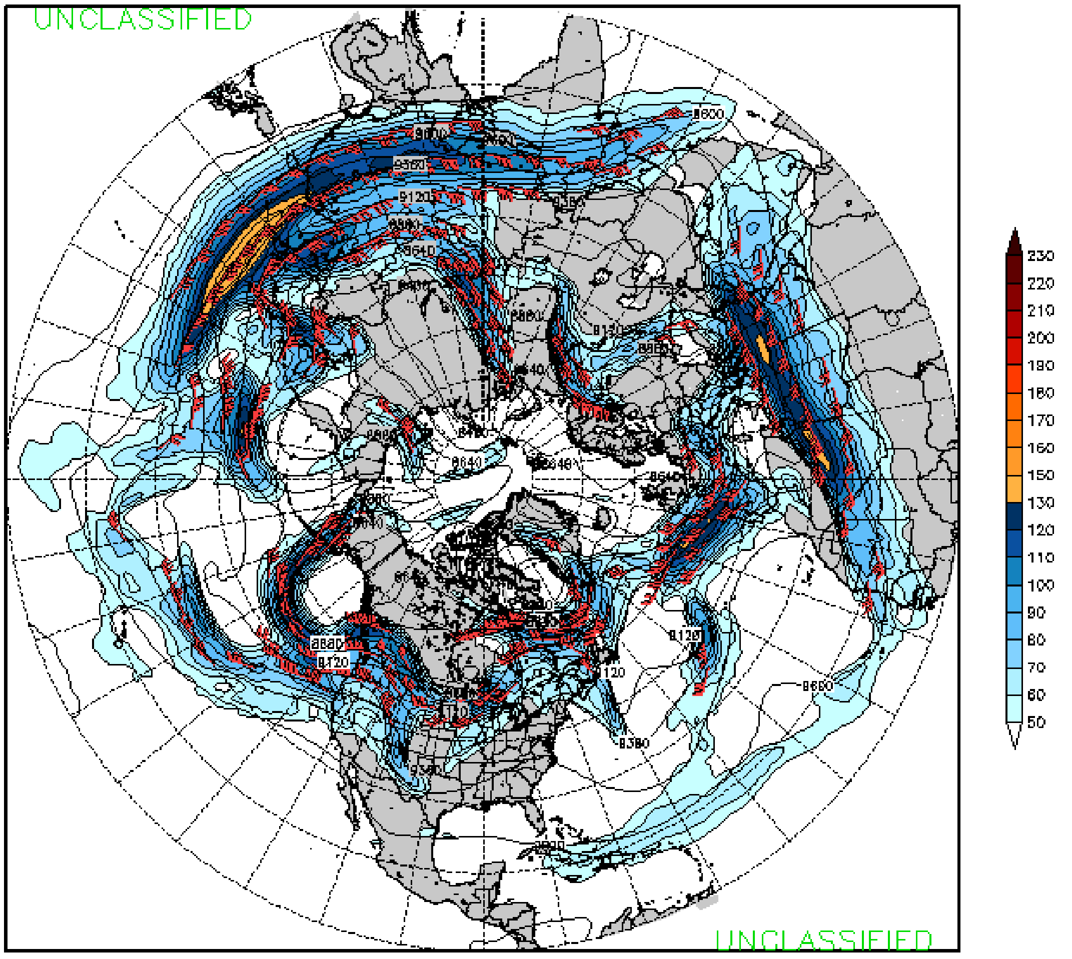
Meridional cross section of longitudinally and time-averaged zonal wind (solid contours, interval of ms^{-1}) and temperature (dashed contours, interval of 5 K) for June-August. Easterly winds are shaded and 0˚ C isotherm is darkened. Wind maxima shown in ms^{-1} , temperature minima shown in ˚C. (NCEP/NCAR reanalyses; after Wallace, 2003.)

Notice significant deviations from zonal symmetry!



Mean zonal wind at the 200-hPa level for December–February averaged for years 1958–1997. Contour interval 10 ms^{-1} (heavy contour, 20 ms^{-1}).
(Based on NCEP/NCAR reanalyses; after Wallace, 2003.)

In addition to its longitudinal dependence, the planetary scale flow also varies from day to day due to its interactions with transient synoptic-scale disturbances.
In fact, observations show that the transient planetary scale flow amplitude is comparable to that of the time-mean. As a result, monthly mean charts tend to smooth out the actual structure of the instantaneous jet-stream since the position and intensity of the jet vary. Thus, at any time the planetary scale flow in the region of the jet-stream has much greater baroclinicity than indicated on time-averaged charts.



VT: Tue 06Z 04 DEC 12
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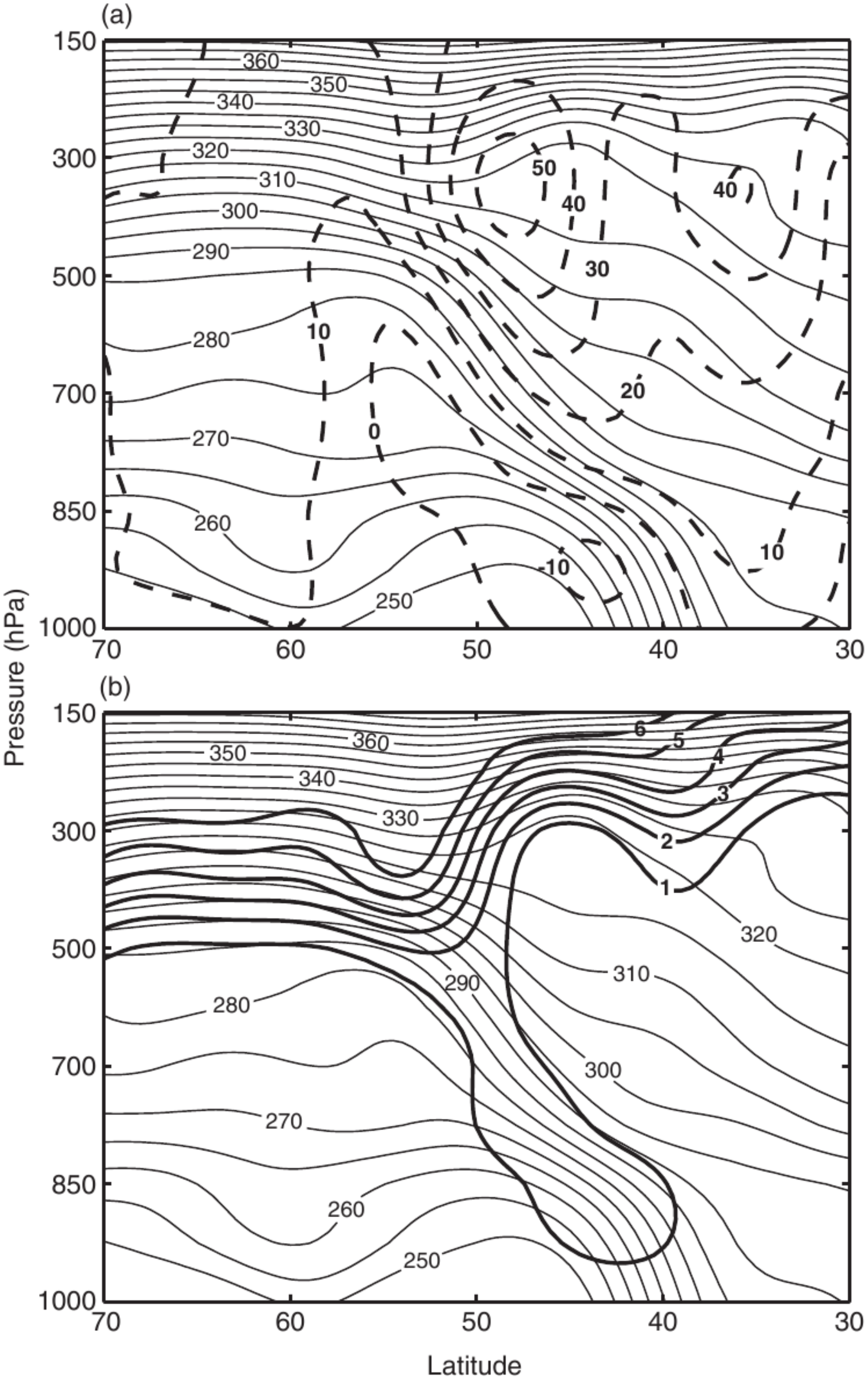


The axis of the jet-stream tends to be located above a narrow sloping zone of strong potential temperature gradients called the polar-frontal zone. The occurrence of an intense jet core above this zone of large magnitude potential temperature gradients is a consequence of the thermal wind balance.

The potential temperature contours illustrate the fact that isentropes (constant θ surfaces) cross the tropopause in the vicinity of the jet so that air can move between the troposphere and the stratosphere without diabatic heating or cooling.

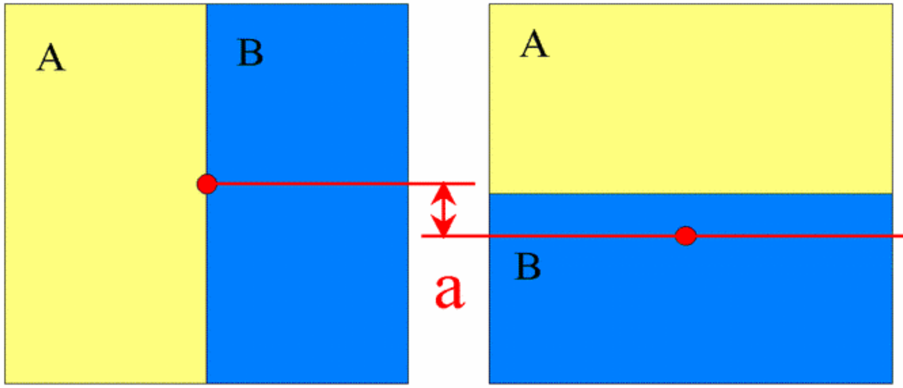
The strong gradient of Ertel potential vorticity at the tropopause, however, provides a strong resistance to cross-tropopause flow along the isentropes.

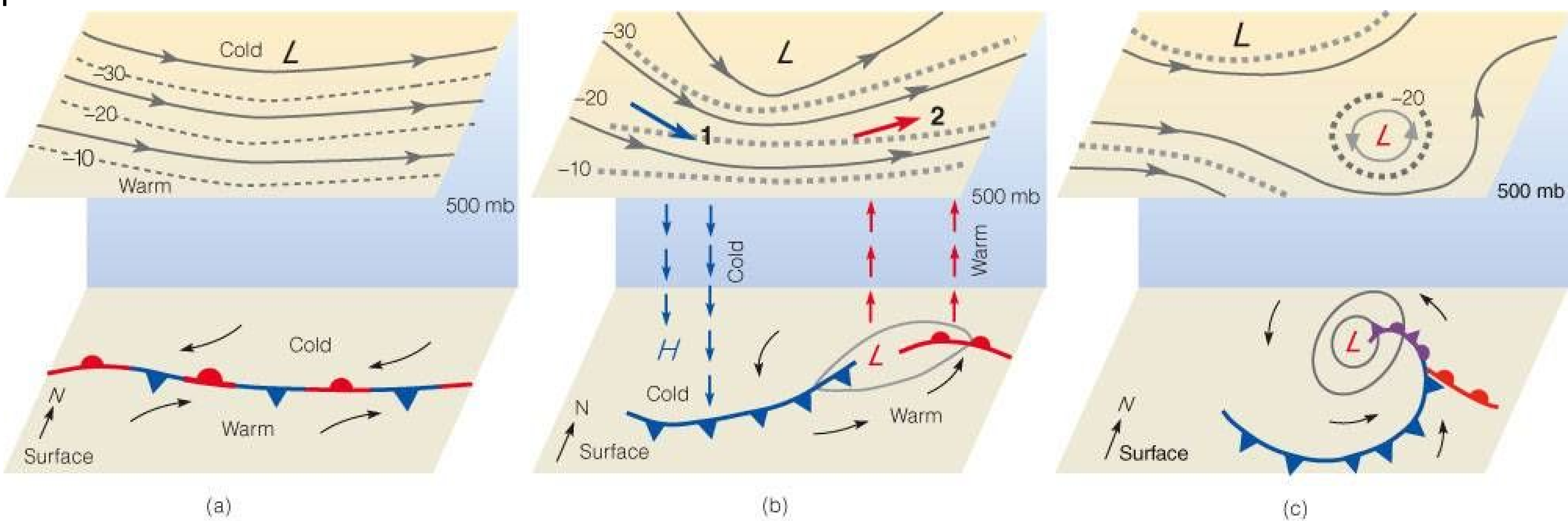
Latitude–height cross sections through a cold front at 80W longitude on 00 UTC January 14, 1999.

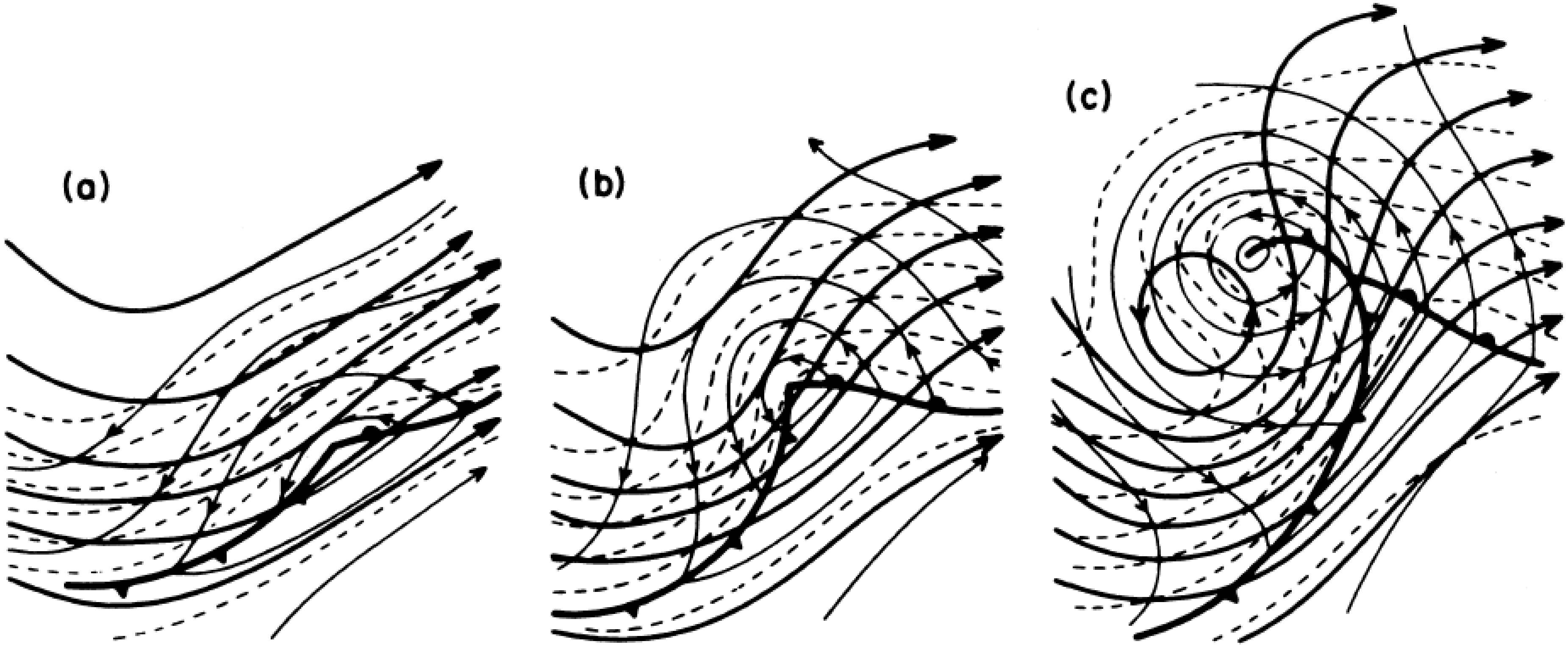
(a) Potential temperature contours (thin solid lines, K) and zona wind isotachs (dashed lines, ms^{-1}).

(b) Thin solid contours as in (a), heavy contours show Ertel potential vorticity labeled in PVU (1PVU = 10^{-6} K kg^{-1} m^{2} s^{-1}).

It is a common observation in fluid dynamics that jets in which strong velocity shears occur may be unstable with respect to small perturbations. By this is meant that any small disturbance introduced into the jet will tend to amplify, drawing energy from the jet as it grows. Most synoptic-scale systems in midlatitudes appear to develop as the result of an instability of the jet-stream flow. This instability, called baroclinic instability, depends on the meridional temperature gradient, particularly at the surface. Hence, through the thermal wind relationship, baroclinic instability depends on vertical shear and tends to occur in the region of the polar frontal zone.

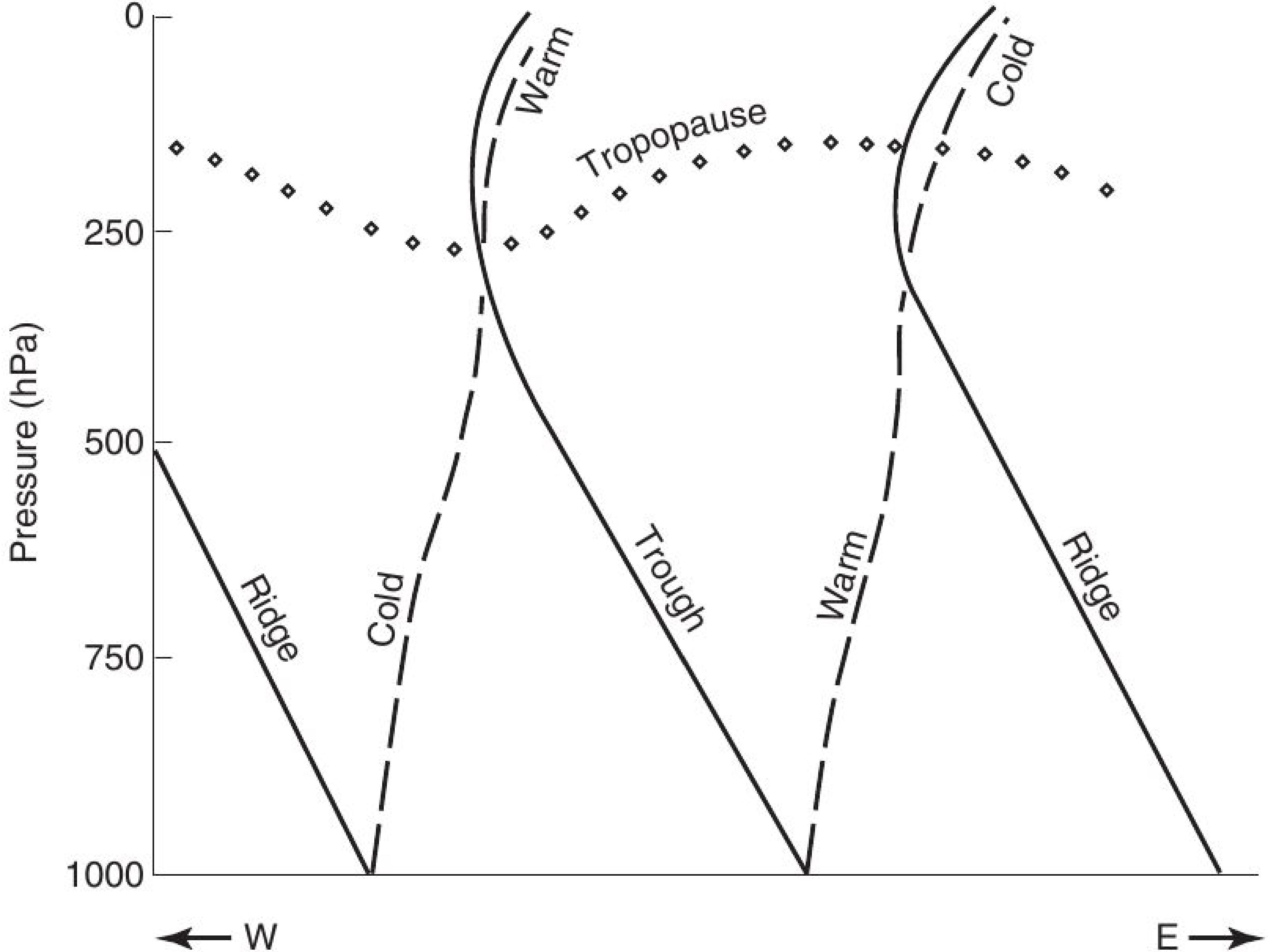




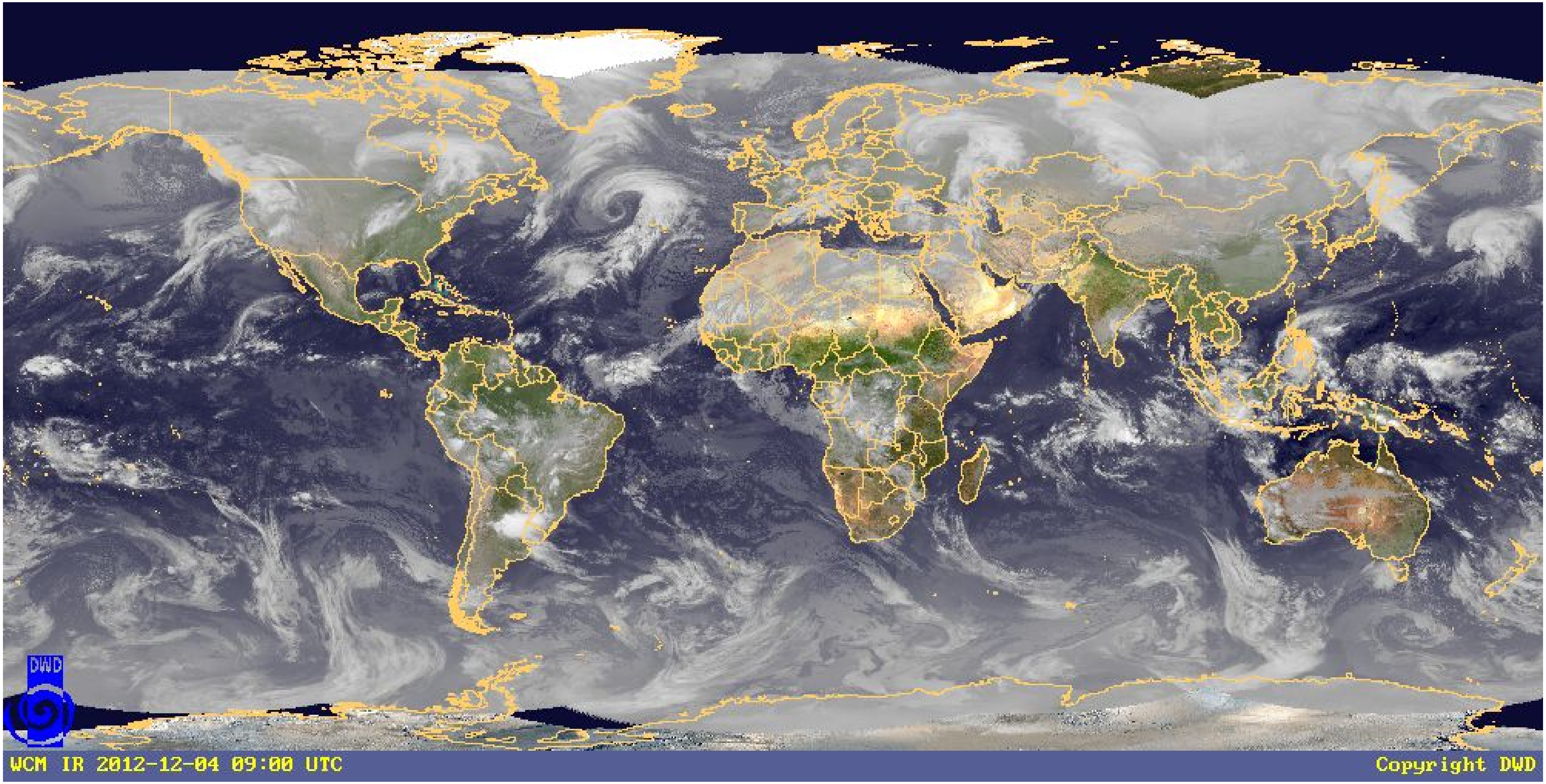


Schematic 500-hPa contours (heavy solid lines), 1000-hPa contours (thin lines), and 1000–500 hPa thickness (dashed) for a developing baroclinic wave at three stages of development. (After Palmer and Newton, 1969.)

In the stage of rapid development of cyclone there is a cooperative interaction between the upper level and surface flows; strong cold advection is seen to occur west of the trough at the surface, with weaker warm advection to the east.
This pattern of thermal advection is a direct consequence of the fact that the trough at 500 hPa lags (lies to the west of) the surface trough so that the mean geostrophic wind in the 1000- to 500-hPa layer is directed across the 1000- to 500-hPa thickness lines toward larger thickness west of the surface trough and toward smaller thickness east of the surface trough.



West–east cross section through a developing baroclinic wave. Solid lines are trough and ridge axes; dashed lines are axes of temperature extrema; the chain of open circles denotes the tropopause.



Baroclinic disturbances around the world..

THE QUASI-GEOSTROPHIC APPROXIMATION
Scale Analysis in Isobaric Coordinates

The dynamical equations in isobaric coordinates are presented here. The horizontal momentum equation, the hydrostatic equation, the continuity equation, and the thermodynamic energy equation may be expressed as:

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k}\times\mathbf{V} = -\boldsymbol{\nabla}\Phi$$

$$\frac{\partial\Phi}{\partial p} = -\alpha = -RT\big/p$$

$$\boldsymbol{\nabla}\bullet\mathbf{V} + \frac{\partial\omega}{\partial p} = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\bullet\boldsymbol{\nabla}\right)T - S_p\omega = J/c_p$$

with the total derivative in the momentum equation defined by

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_p + (\mathbf{V}\bullet\boldsymbol{\nabla})_p + \omega\frac{\partial}{\partial p}$$

where $\omega \equiv Dp/Dt$ is the pressure change following the motion, $S_p \equiv -T(\partial \ln\theta / \partial p)$ is the static stability parameter [$S_p \approx 5\times10^{-4}\ K\ Pa^{-1}$ in the midtroposphere], J is heating rate.

Let's separate the horizontal velocity into geostrophic and ageostrophic parts by letting $\mathbf{V} = \mathbf{V}_g + \mathbf{V}_a$ where the geostrophic wind is defined as $\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi$

and the ageostrophic wind, $\mathbf{V}_a$, is just the difference between the total horizontal wind and the geostrophic wind. We have here assumed that the meridional length scale, L, is small compared to the radius of the earth so that the geostrophic wind may be defined using a constant reference latitude value of the Coriolis parameter.
For the systems of interest $|\mathbf{V}_g| \gg |\mathbf{V}_a|$, more precisely, $|\mathbf{V}_a| / |\mathbf{V}_g| \sim O(\mathrm{Ro})$

The ratio of the magnitudes of the ageostrophic and geostrophic winds is the same order of magnitude as the Rossby number.

The momentum can then be approximated to O(Ro) by its geostrophic value, and the rate of change of momentum or temperature following the horizontal motion can be approximated to the same order by the rate of change following the geostrophic wind. Thus, $\mathbf{V}$ can be replaced by $\mathbf{V}_g$ and the vertical advection, which arises only from the ageostrophic flow, can be neglected.

The rate of change of momentum following the total motion is then approximately equal to the rate of change of the geostrophic momentum following the geostrophic wind:

$$\frac{D\mathbf{V}}{Dt} \approx \frac{D_g \mathbf{V}_g}{Dt}$$

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

Let's use beta-plane approximation: $f = f_0 + \beta y$

$$\beta \equiv (df/dy)_{\phi_0} = 2\Omega \cos\phi_0 / a \text{ and } y = 0 \text{ at } \phi_0$$

For synoptic-scale motions, the ratio of the first two terms in the expansion of f has an order of magnitude:

$$\frac{\beta L}{f_0} \sim \frac{\cos\phi_0}{\sin\phi_0}\frac{L}{a} \sim \mathrm{O}(\mathrm{Ro}) \ll 1$$

The acceleration following the motion is equal to the difference between the Coriolis force and the pressure gradient force. This difference depends on the departure of the actual wind from the geostrophic wind. Thus, it is not permissible to simply replace the horizontal velocity by its geostrophic value. We write:

$$f\mathbf{k}\times\mathbf{V} + \boldsymbol{\nabla}\Phi = (f_0 + \beta y)\,\mathbf{k}\times\left(\mathbf{V}_g + \mathbf{V}_a\right) - f_0\mathbf{k}\times\mathbf{V}_g$$
$$\approx f_0\mathbf{k}\times\mathbf{V}_a + \beta y\mathbf{k}\times\mathbf{V}_g$$

The approximate horizontal momentum equation thus has the form:

$$\frac{D_g\mathbf{V}_g}{Dt} = -f_0\mathbf{k}\times\mathbf{V}_a - \beta y\mathbf{k}\times\mathbf{V}_g$$

In the above each term is $\mathrm{O}(\mathrm{Ro})$ compared to the pressure gradient force, whereas terms neglected are $\mathrm{O}(\mathrm{Ro}^2)$ or smaller.

The geostrophic wind is nondivergent:

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}_a = \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$$

which gives the continuity equation in the form:

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Indicating that ω is determined only by the ageostrophic part of the wind field.

In the thermodynamic energy equation the horizontal advection can be approximated by its geostrophic value. However, the vertical advection is not neglected, but forms part of the adiabatic heating and cooling term, despite the smallness of the vertical velocity.
The adiabatic heating and cooling term can be simplified by dividing the total temperature field, into a basic state (standard atmosphere) plus a deviation from the basic state:,

$$T_{tot}(x, y, p, t) = T_0(p) + T(x, y, p, t)$$

Because $|dT_0/dp| \gg |\partial T/\partial p|$ only the basic state portion of the temperature field need be included in the static stability term, and the quasi-geostrophic thermodynamic energy equation may be expressed in the form:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right) T - \left(\frac{\sigma p}{R}\right) \omega = \frac{J}{c_p} \qquad \sigma \equiv -RT_0 p^{-1} d\ln\theta_0/dp$$

$$\sigma \approx 2.5 \times 10^{-6} \mathrm{m}^2\, \mathrm{Pa}^{-2} \mathrm{s}^{-2}$$

$$\frac{\partial \Phi}{\partial p} = -\alpha = -RT/p$$

The alternative form of the thermodynamic energy equation can be expressed in terms of the geopotential field:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \bullet \nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\omega = \frac{\kappa J}{p} \qquad \kappa \equiv R/c_p$$

This is the last equation from the set of quasi-geostrophic equations, the others (from the previous slides) are:

$$\mathbf{V}_g \equiv f_0^{-1}\mathbf{k} \times \nabla\Phi$$

$$\frac{D_g \mathbf{V}_g}{Dt} = -f_0 \mathbf{k} \times \mathbf{V}_a - \beta y \mathbf{k} \times \mathbf{V}_g$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

These form a complete set in the dependent variables, $\mathbf{V}_g$, $\mathbf{V}_a$, and ω (provided that the diabatic heating rate is known). This form of the equations is not, however, suitable as a prediction system. It is useful to replace momentum equation by an equation for the evolution of the vorticity of the geostrophic wind, in which case only the divergent part of the ageostrophic wind plays a role in the dynamics.

The Quasi-Geostrophic Vorticity Equation

The vertical component of vorticity can be approximated geostrophically:

$$f_0 v_g = \frac{\partial \Phi}{\partial x}, \quad f_0 u_g = -\frac{\partial \Phi}{\partial y}$$

$$\zeta_g = \mathbf{k} \cdot \nabla \times \mathbf{V}_g$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla^2 \Phi$$

The above equation can be used to determine $\zeta_g (x, y)$ from a known field $\Phi(x, y)$. Alternatively, it can be solved by inverting the Laplacian operator to determine from a known distribution of ζ_g, provided that suitable conditions on Φ are specified on the boundaries of the region in question.

This invertibility is one reason why vorticity is such a useful forecast diagnostic; if the evolution of the vorticity can be predicted, then inversion of the equation yields the evolution of the geopotential field, from which it is possible to determine the geostrophic wind and temperature distributions.

Since the Laplacian of a function tends to be a maximum where the function itself is a minimum, positive vorticity implies low values of geopotential and vice versa.

The quasi-geostrophic vorticity equation can be obtained from the x and y components of the quasi-geostrophic momentum equation:

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = 0$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = 0$$

Taking spatial derivatives and using the fact that the divergence of the geostrophic wind vanishes, yields the vorticity equation:

$$\frac{D_g \zeta_g}{Dt} = -f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g$$

$$\mathrm{D}_g f / Dt = \mathbf{V}_g \bullet \boldsymbol{\nabla} f = \beta v_g$$

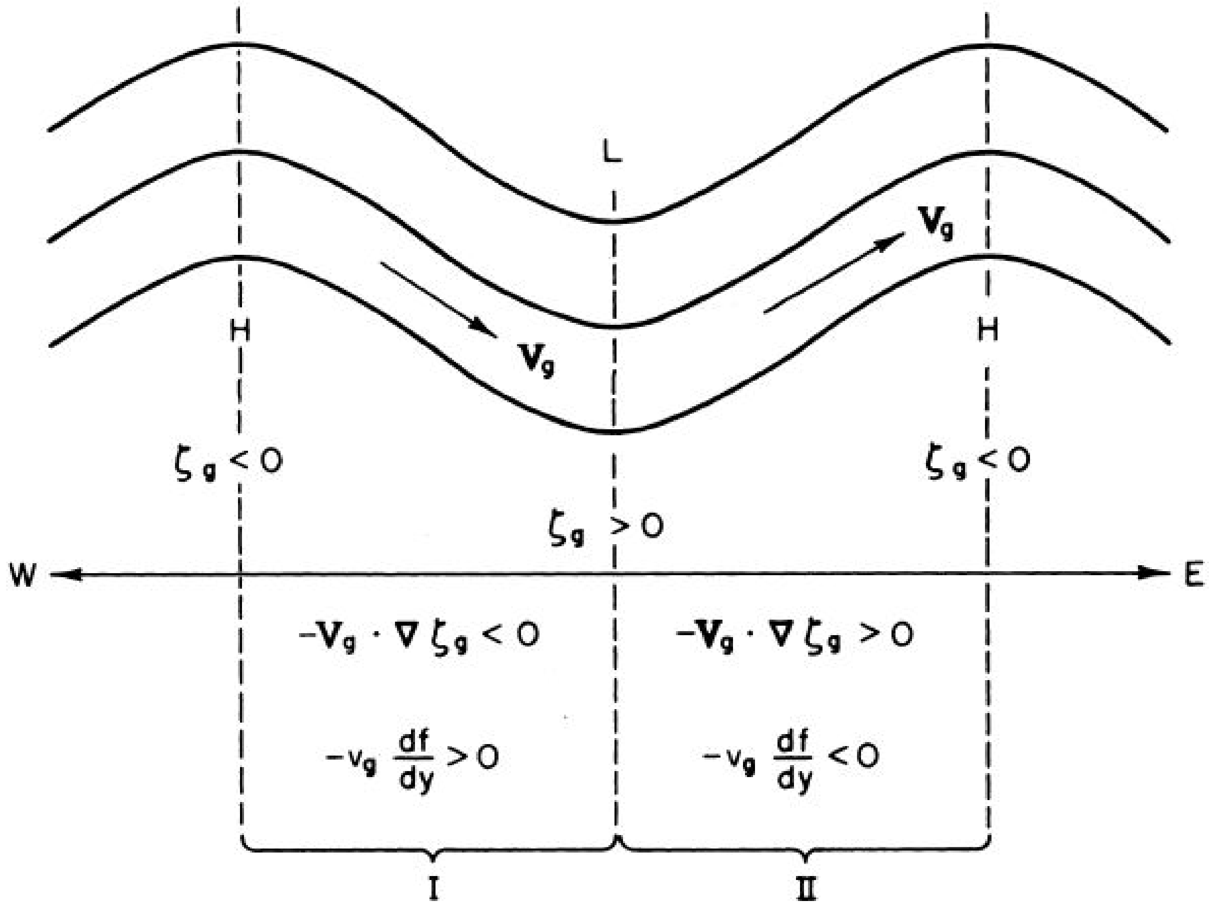
$$\frac{\partial \zeta_g}{\partial t} = -\mathbf{V}_g \bullet \boldsymbol{\nabla} \left(\zeta_g + f \right) + f_0 \frac{\partial \omega}{\partial p}$$

which states that the local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns (the divergence effect).

The vorticity tendency due to vorticity advection may be rewritten as:

$$-\mathbf{V}_g \cdot \boldsymbol{\nabla}\left(\zeta_g + f\right) = -\mathbf{V}_g \cdot \boldsymbol{\nabla}\zeta_g - \beta v_g$$

The two terms on the right represent the geostrophic advections of relative vorticity and planetary vorticity, respectively. For disturbances in the westerlies, these two effects tend to have opposite signs, as illustrated schematically:



In order to investigate details of vorticity advection consider geopotential in sinusoidal form:

$$\Phi(x, y) = \Phi_0 - f_0 U y + f_0 A \sin kx \cos ly \qquad y = a(\phi - \phi_0)$$

The parameters Φ_0 , U , and A depend only on pressure, and the wave numbers k and l are defined as $k = 2\pi/Lx$ and $l = 2\pi/Ly$ with Lx , Ly the wavelengths in the x and y directions, respectively.
The geostrophic wind components are then given by

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} = U + u'_g = U + lA \sin kx \sin ly$$
$$v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} = v'_g = +kA \cos kx \cos ly$$

and $\left(u'_g,\ v'_g\right)$ is the geostrophic wind due to the synoptic wave disturbance. Then

$$\zeta_g = f_0^{-1}\,\nabla^2\Phi = -\left(k^2 + l^2\right) A \sin kx\ \cos ly$$

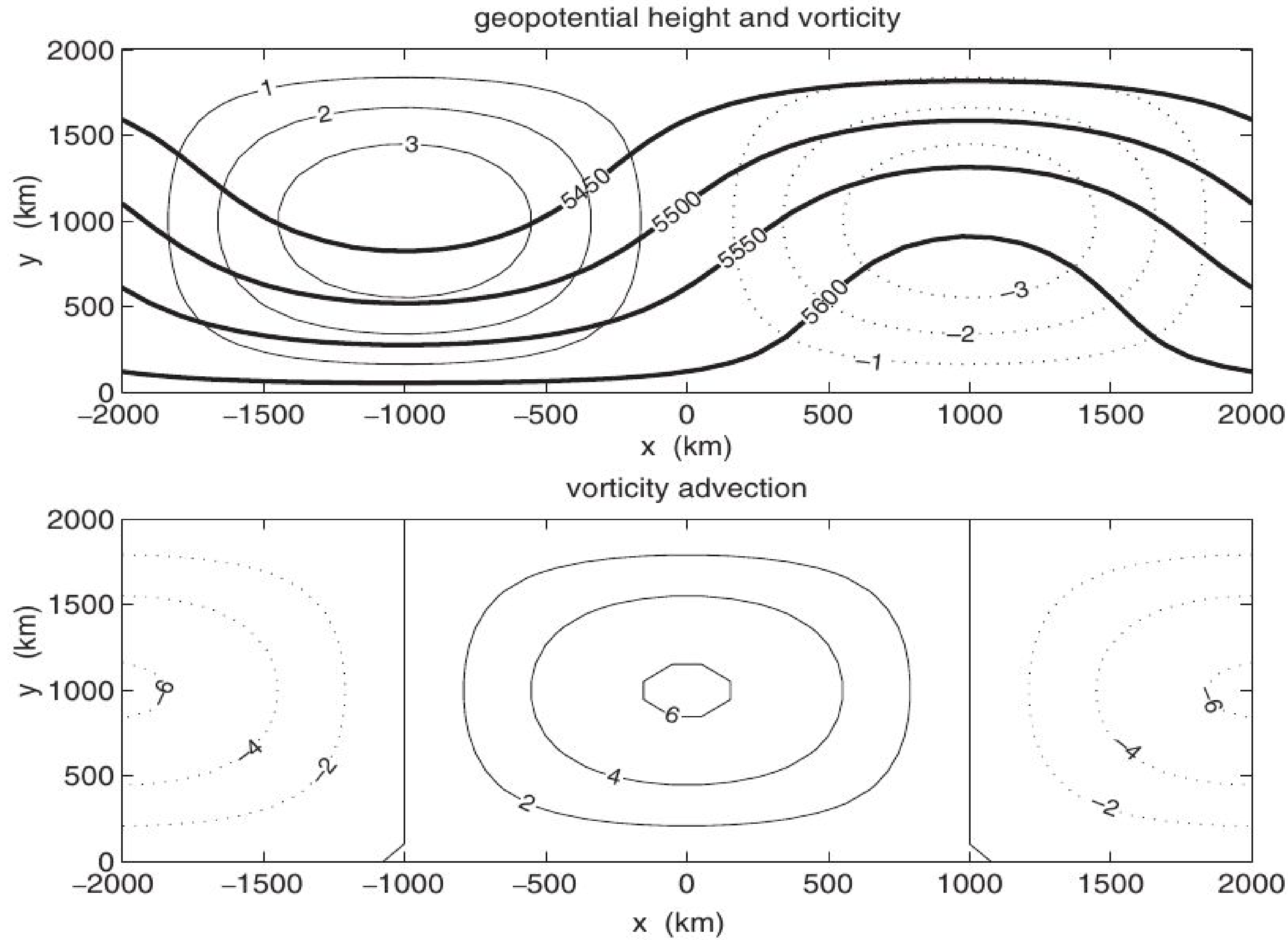
It can be shown that in this simple case the advection of relative vorticity by the wave component of the geostrophic wind vanishes: $u'_g \partial\zeta_g \big/ \partial x + \nu'_g \partial\zeta_g \big/ \partial y = 0$

and the advection of relative vorticity is:

$$-u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} = -U \frac{\partial \zeta_g}{\partial x} = +kU\left(k^2 + l^2\right) A \cos kx \cos ly$$

Consequently the advection of planetary vorticity can be expressed as

$$-\beta v_g = -\beta k A \cos kx \cos ly$$



(Top) Geopotential height in units of m, and relative vorticity in units of 10^{-5} s^{-1} for the sinusoidal disturbance of equation (6.20). Here $\Phi_0 = 5.5 \times 10^4$ m^2 s^{-2}, $f_0 = 10^{-4}$ s^{-1}, f_0A $=$ 800 m^2 s^{-2}, $U = 10$ m s^{-1}, and $k = l = (\pi/2) \times 10^{-6}$ m^{-1}. (Bottom) Advection of relative vorticity in units of 10^{-10} s^{-2} for the disturbance shown above.

QUASI-GEOSTROPHIC PREDICTION

The evolution of the geostrophic circulation can actually be determined without explicitly determining the distribution of ω. Defining the geopotential tendency $\chi \equiv \partial\Phi/\partial t$, and recalling that the order of partial differentiation may be reversed, the geostrophic vorticity equation can be expressed as

$$\frac{1}{f_0}\nabla^2\chi = -\mathbf{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

An analogous equation dependent on these two variables can be obtained from the thermodynamic energy equation by multiplying through by f_0/σ and differentiating with respect to p. Using the definition of χ , the result can be expressed as

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -\frac{\partial}{\partial p}\left[\frac{f_0}{\sigma}\mathbf{V}_g \cdot \nabla\left(\frac{\partial\Phi}{\partial p}\right)\right] - f_0\frac{\partial\omega}{\partial p} - f_0\frac{\partial}{\partial p}\left(\frac{\kappa J}{\sigma p}\right)$$

The left side of can be interpreted as the local rate of change of a normalized static stability anomaly:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial\chi}{\partial p}\right) = -Rf_0\frac{\partial}{\partial p}\left(\frac{1}{\sigma p}\frac{\partial T}{\partial t}\right) = -f_0\frac{\partial}{\partial p}\left(\frac{1}{S_p}\frac{\partial T}{\partial t}\right) \approx -\frac{\partial}{\partial t}\left(\frac{f_0}{S_p}\frac{\partial T}{\partial p}\right)$$

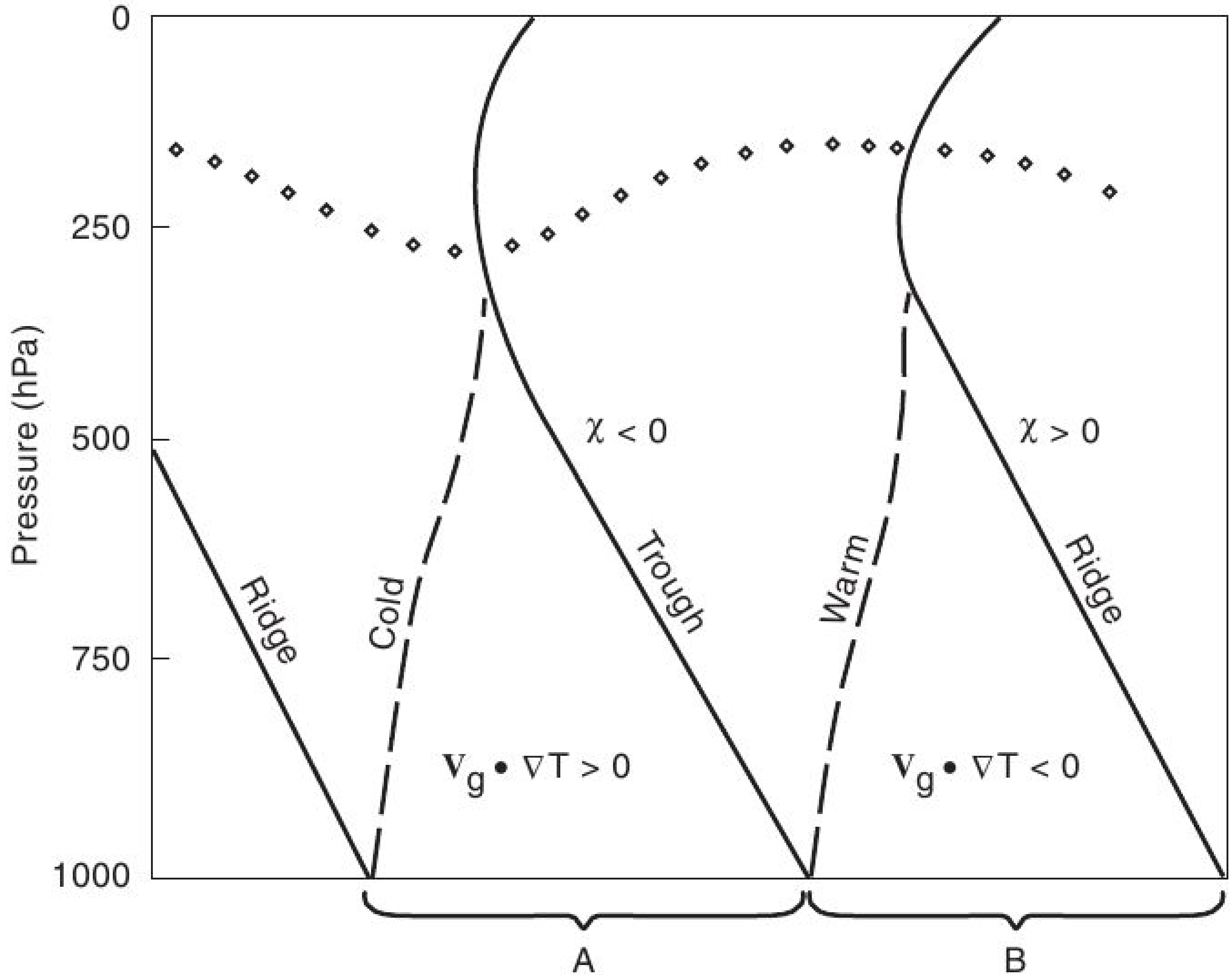
$$\frac{\partial \Phi}{\partial p} = -\alpha = -RT/p$$

Geopotential Tendency

If for simplicity we set J = 0 and eliminate ω between the equations on the previous slide to obtain an equation that determines the local rate of change of geopotential in terms of the three-dimensional distribution of the geopotential field:

$$\underbrace{\left[\nabla^2 + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{A} = \underbrace{-f_0 \mathbf{V}_g \bullet \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)}_{B} \underbrace{- \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\mathbf{V}_g \bullet \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{C}$$

This equation is often referred to as the geopotential tendency equation. It provides a relationship between the local geopotential tendency (term A) and the distributions of vorticity advection (term B) and thickness advection (term C). If the distribution of is known at a given time, then terms B and C may be regarded as known forcing functions, and yje above is a linear partial differential equation in the unknown χ.



East–west section through a developing synoptic disturbance showing the relationship of temperature advection to the upper level height tendencies. A and B designate, respectively, regions of cold advection and warm advection in the lower troposphere.