Dynamics of the Atmosphere and the Ocean

Lecture 6

Szymon Malinowski

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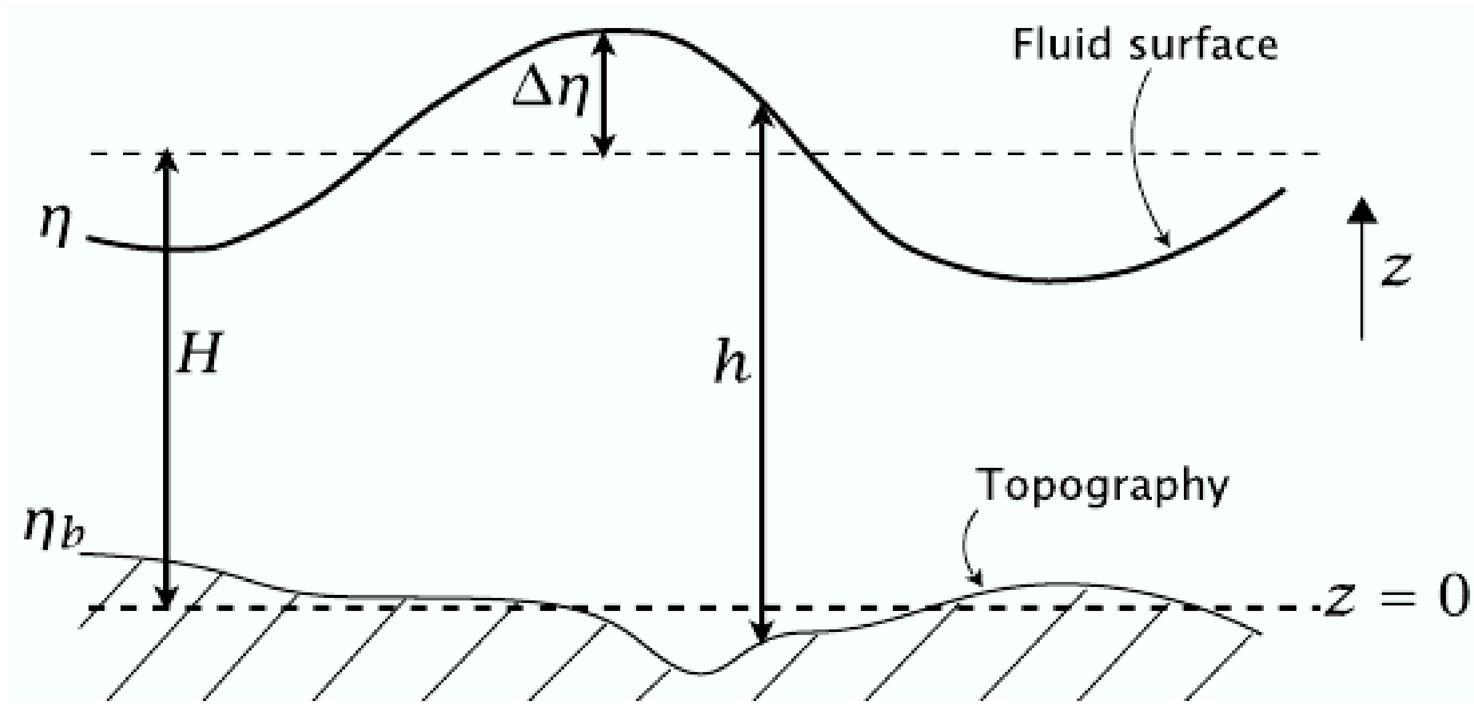


Fig. 3.1 A shallow water system. $h(x, y)$ is the thickness of a water column, H its mean thickness, $\eta(x, y)$ the height of the free surface and η_b is the height of the lower, rigid, surface, above some arbitrary origin, typically chosen such that the average of η_b is zero. $\Delta\eta$ is the deviation free surface height, so we have $\eta = \eta_b + h = H + \Delta\eta$.

3.1.1 Momentum equations

The vertical momentum equation is just the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g, \tag{3.1}$$

and, because density is assumed constant, we may integrate this to

$$p(x, y, z) = -\rho g z + p_o \tag{3.2}$$

At the top of the fluid, $z = \eta$, the pressure is determined by the weight of the overlying fluid and this is assumed negligible. Thus, $p = 0$ at $z = \eta$ giving

$$p(x, y, z) = \rho g (h(x, y) - z) \tag{3.3}$$

The consequence of this is that the horizontal gradient of pressure is independent of height. That is

$$\nabla_z p = \rho g \nabla_z \eta \tag{3.4}$$

where

$$\nabla_z = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \tag{3.5}$$

The velocities u and v are functions only of x, y and t and the horizontal momentum equation is therefore

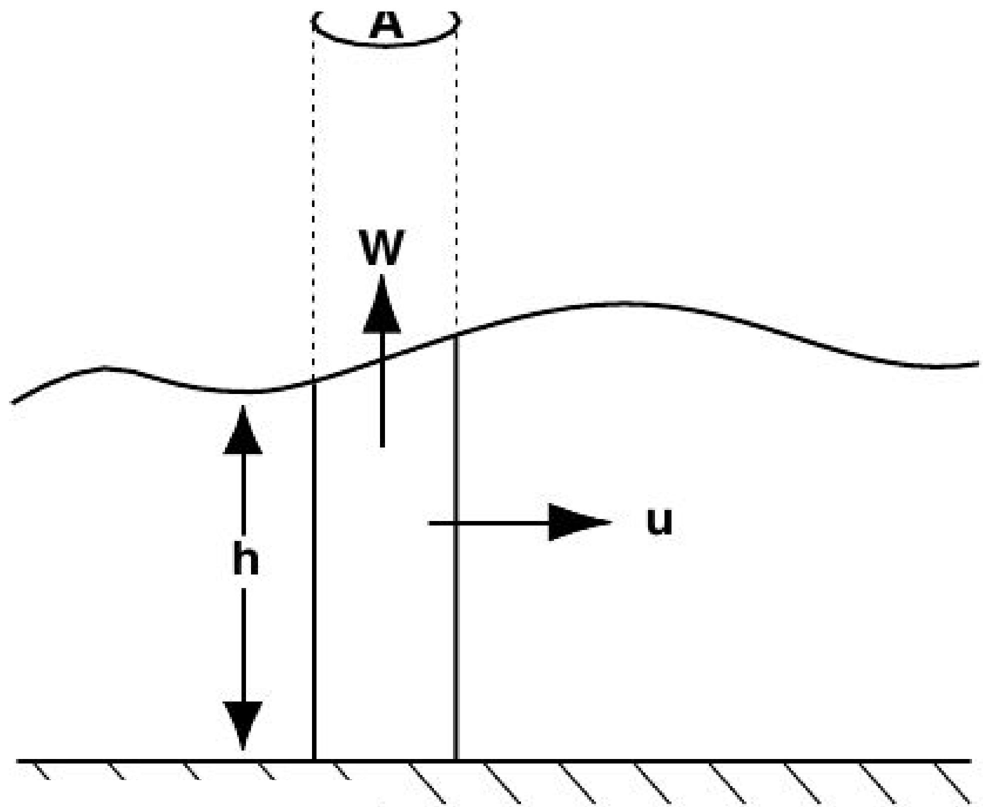
$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} = -g\nabla\eta. \tag{3.7}$$

In the presence of rotation horizontal equation of motion easily generalizes to:

$$\boxed{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta}, \tag{3.8}$$

where $\boldsymbol{f} = f\mathbf{k}$. Just as with the primitive equations, f may be constant or may vary with latitude, so that on a spherical planet $f = 2\Omega \sin\vartheta$ and on the β-plane $f = f_0 + \beta y$.

Figure 3.2 The mass budget for a column of area A in a shallow water system. The fluid leaving the column is $\oint \rho h \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}l$ where $\boldsymbol{n}$ is the unit vector normal to the boundary of the fluid column. There is a non-zero vertical velocity at the top of the column if the mass convergence into the column is non-zero.



$$F_m = \text{Mass flux in} = -\int_S \rho\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = -\oint \rho h \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}l = -\int_A \nabla \cdot (\rho \boldsymbol{u} h)\, \mathrm{d}A.$$

On the other hand mass flux can be written as:

$$F_m = \frac{\mathrm{d}}{\mathrm{d}t} \int \rho \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_A \rho h \, \mathrm{d}A = \int_A \rho \frac{\partial h}{\partial t} \, \mathrm{d}A$$

Comparing two expressions for mass flux one gets conservation of mass in the following form:

$$\int_A \left[\frac{\partial}{\partial t} h + \nabla \cdot (\boldsymbol{u} h) \right] \mathrm{d}A = 0$$

the area is arbitrary the integrand itself must vanish, resulting in

$$\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u} h) = 0 \qquad \frac{\mathrm{D}h}{\mathrm{D}t} + h \nabla \cdot \boldsymbol{u} = 0$$

There are many ways to derive the above.

Momentum equations (hydrostatic balance + horizontal momentum) together with the mass conservation form the simplest set of equations applicable to geophysicsl fluid dynamics: shallow water equations.

The Shallow Water Equations

For a single-layer fluid, and including the Coriolis term, the inviscid shallow water equations are:

Momentum:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta. \tag{SW.1}$$

Mass Conservation:

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0 \qquad \text{or} \qquad \frac{\partial h}{\partial t} + \nabla\cdot(h\boldsymbol{u}) = 0. \tag{SW.2}$$

where $\boldsymbol{u}$ is the horizontal velocity, h is the total fluid thickness, η is the height of the upper free surface and η_b is the height of the lower surface (the bottom topography). Thus, $h(x, y, t) = \eta(x, y, t) - \eta_b(x, y)$. The material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \tag{SW.3}$$

with the rightmost expresion holding in Cartesian coordinates.

Because the horizontal velocity is depth independent, the vertical velocity plays no role in advection. Vertical velocity is certainly not zero for but because of the vertical independence of the horizontal flow w does have a simple vertical structure;

$$\frac{\partial w}{\partial z} = -\nabla \cdot \boldsymbol{u}$$

which after integration really gives w independent of height:

$$w = w_b - (\nabla \cdot \boldsymbol{u})(z - \eta_b).$$

$$\frac{Dz}{Dt} = \frac{D\eta_b}{Dt} - (\nabla \cdot \boldsymbol{u})(z - \eta_b),$$

at the upper surface $w = D\eta/Dt$ so that here we have

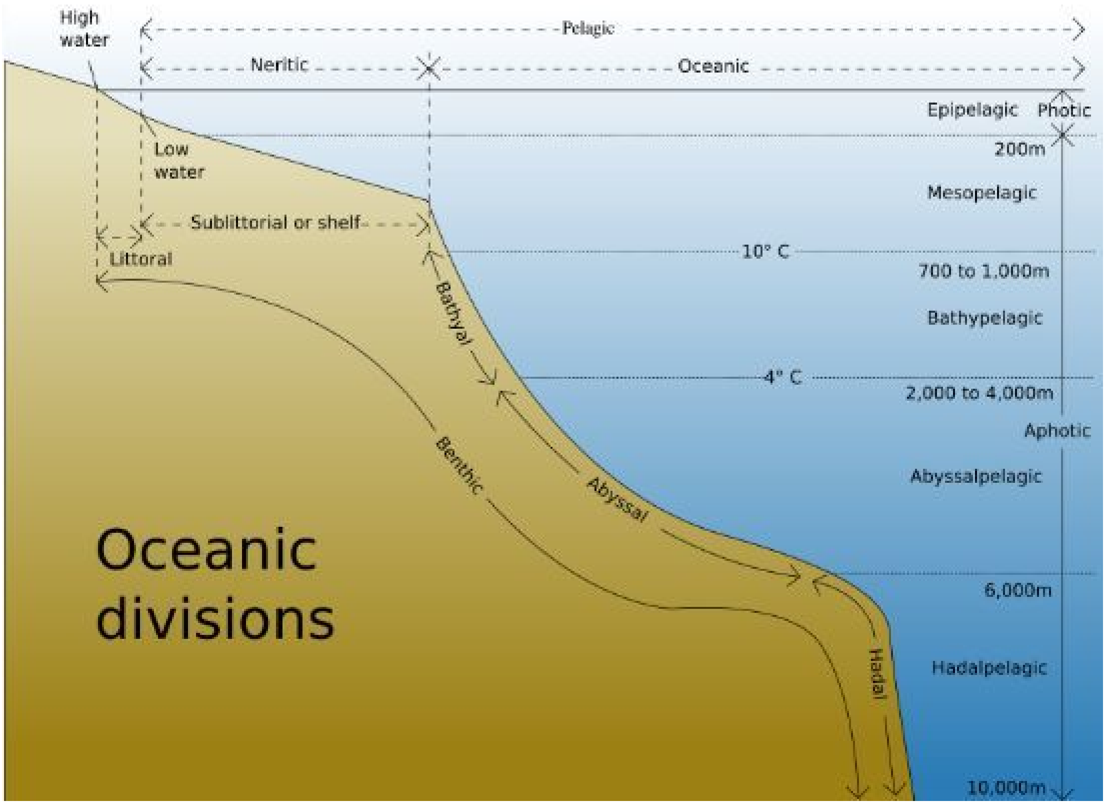
$$\frac{D\eta}{Dt} = \frac{D\eta_b}{Dt} - (\nabla \cdot \boldsymbol{u})(\eta - \eta_b),$$

Eliminating the divergence term from the last two equations gives

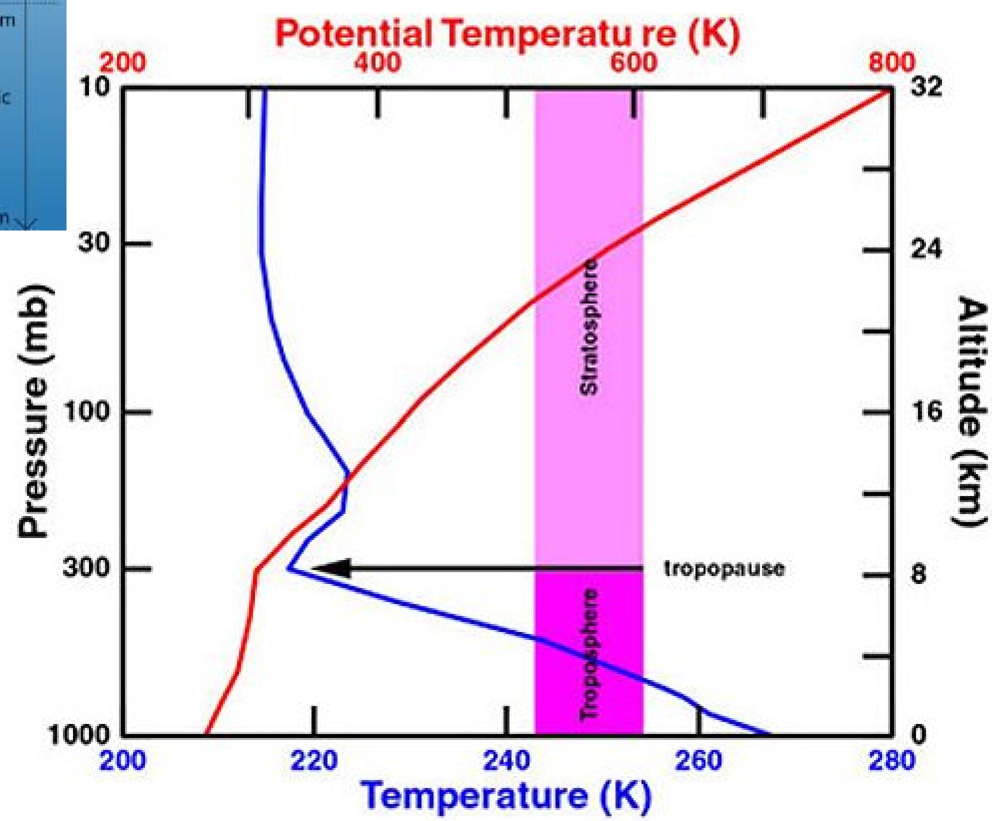
$$\frac{D}{Dt}(z - \eta_b) = \frac{z - \eta_b}{\eta - \eta_b} \frac{D}{Dt}(\eta - \eta_b),$$

$$\frac{D}{Dt}\left(\frac{z - \eta_b}{\eta - \eta_b}\right) = \frac{D}{Dt}\left(\frac{z - \eta_b}{h}\right) = 0.$$

REDUCED GRAVITY EQUATIONS



Consider now a single shallow moving layer of fluid on top a deep, quiescent fluid layer and beneath a fluid of negligible inertia. This configuration is often used a model of the upper ocean: the upper layer represents flow in perhaps the upper few hundred meters of the ocean, the lower layer the near-stagnant abyss.

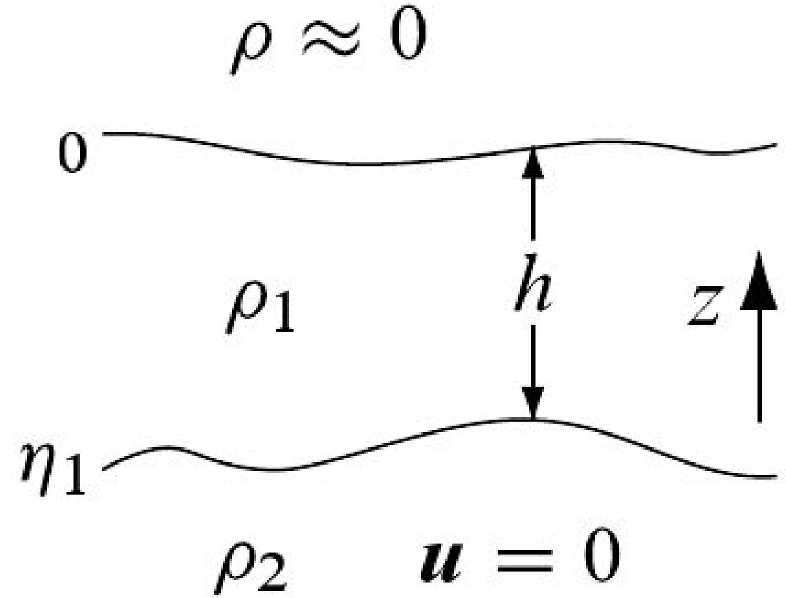


If we turn the model upside-down we have a model, perhaps slightly less realistic, of the atmosphere: the lower layer represents motion in the troposphere above which lies an inactive stratosphere. The equations of motion are virtually the same in both cases.

Pressure gradient in the active layer

We'll derive the equations for the oceanic case (active layer on top)

Free upper surface:



$$p_1(z) = g\rho_1(\eta_0 - z),$$

$$\frac{1}{\rho_1}\nabla p_1 = -g\nabla\eta_0,$$

The above gives momentum equation in the form:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -g\nabla\eta_0.$$

In the lower layer:

$$p_2(z) = \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z).$$

Since the layer is motionless the horizontal pressure gradient in it is zero :

$$\rho_1 g\eta_0 = -\rho_1 g'\eta_1 + \text{constant},$$

Defining “reduced gravity” as: $g' = g(\rho_2 - \rho_1)/\rho_1$

we get the following momentum equation:

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = g' \nabla \eta_1 .$$

the mass conservation equation has the form:

$$\frac{Dh}{Dt} + h \nabla \cdot \boldsymbol{u} = 0,$$

where $h = \eta_0 - \eta_1 .$

Since g>>g' , surface displacements are much smaller than the displacements at the interior interface. This sis the case of the real ocean where the mean interior isopycnal displacements may be several tens of meters but variations in the mean height of ocean surface are of order centimeters.

The smallness of the upper surface displacement suggests that we will make little error is we impose a rigid lid at the top of the fluid. Displacements are no longer allowed, but the lid will in general impart a pressure force to the fluid.

The rigid lid approximation

Suppose that this is P(x; y; t) is the pressure at the ocean surface. Then the horizontal pressure gradient in the upper layer is :

$$\nabla p_1 = \nabla P.$$

The pressure in the lower layer is given by hydrostasy:

$$\begin{aligned} p_2 &= -\rho_1 g \eta_1 + \rho_2 g(\eta_1 - z) + P \\ &= \rho_1 g h - \rho_2 g(h + z) + P, \end{aligned}$$

$$\nabla p_2 = -g(\rho_2 - \rho_1)\nabla h + \nabla P,$$

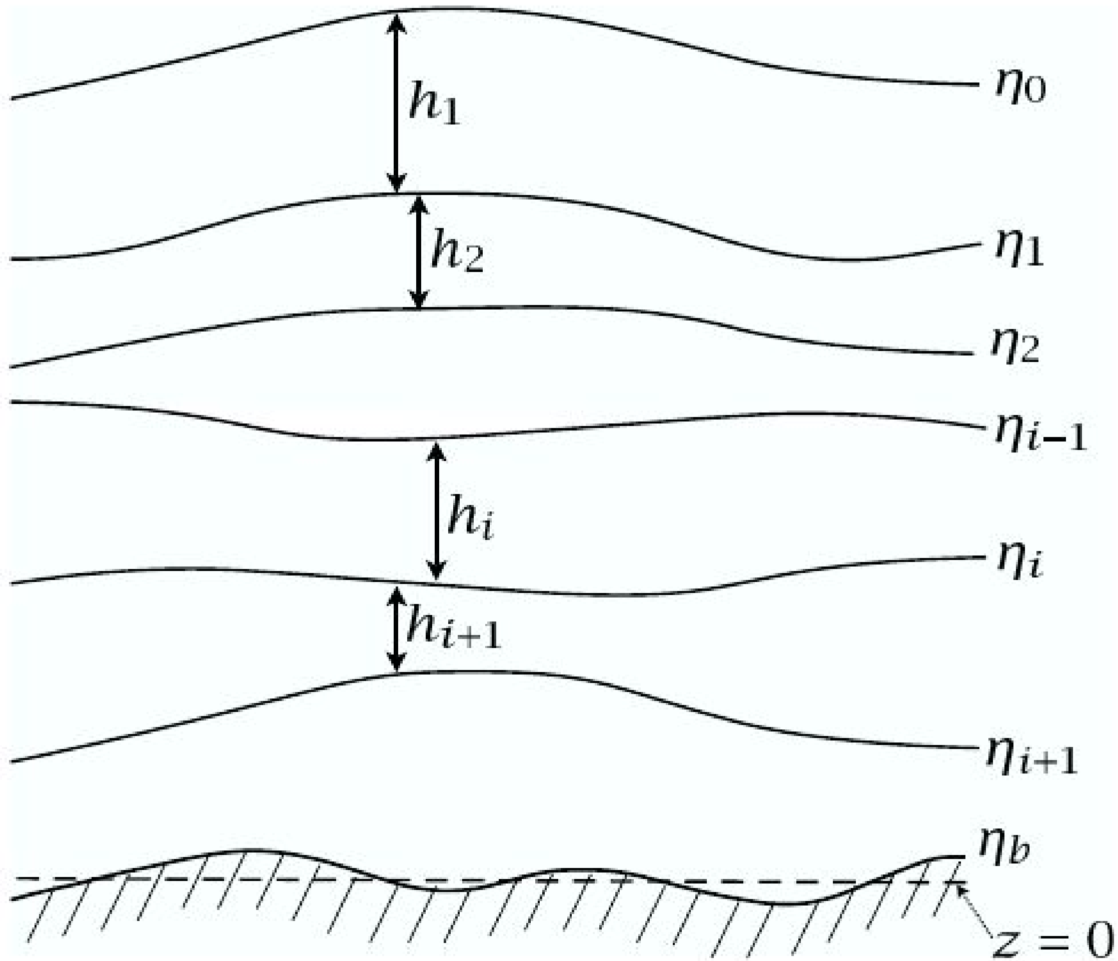
For zero gradient on p_2 the above takes the form: $g(\rho_2 - \rho_1)\nabla h = \nabla P$
Which gives the momentum equation:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g_1' \nabla h.$$

In the above $g' = g(\rho_2 - \rho_1)/\rho_1$ which indicates that density difference between the two layers is important.

MULTI-LAYER SHALLOW WATER EQUATIONS

Figure 3.4 The multi-layer shallow water system. The layers are numbered from the top down. The coordinates of the interfaces are denoted η, and the layer thicknesses h, so that $h_i = \eta_i - \eta_{i-1}$.



We now consider multiple layers of fluid stacked on top of each other.
This is a crude representation of continuous stratification, but it turns out to be a powerful model of many geophysically interesting phenomena. The pressure is continuous across the interface, but the density jumps discontinuously and this allows the horizontal velocity to have a corresponding discontinuity.

Pressure is given by the hydrostatic approximation. Anywhere the can be find by integrating down from the top.

At a height z in the first layer we have: $p_1 = \rho_1 g(\eta_0 - z),$

and in the following layer $p_2 = \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z) = \rho_1 g\eta_0 + \rho_0 g_1'\eta_1 - \rho_2 g z,$

With $g_1' = g(\rho_2 - \rho_1)/\rho_1$. Such a reasoning can be extended into next layers:

$$p_n = \rho_1 \sum_{i=0}^{n-1} g_i' \eta_i,$$

where $g_i' = g(\rho_{i+1} - \rho_i)/\rho_1$ (taking $\rho_0 = 0$).

The above can be written in terms of the layer thicknesses:

$$\eta_n = \eta_b + \sum_{i=n+1}^{i=N} h_i.$$

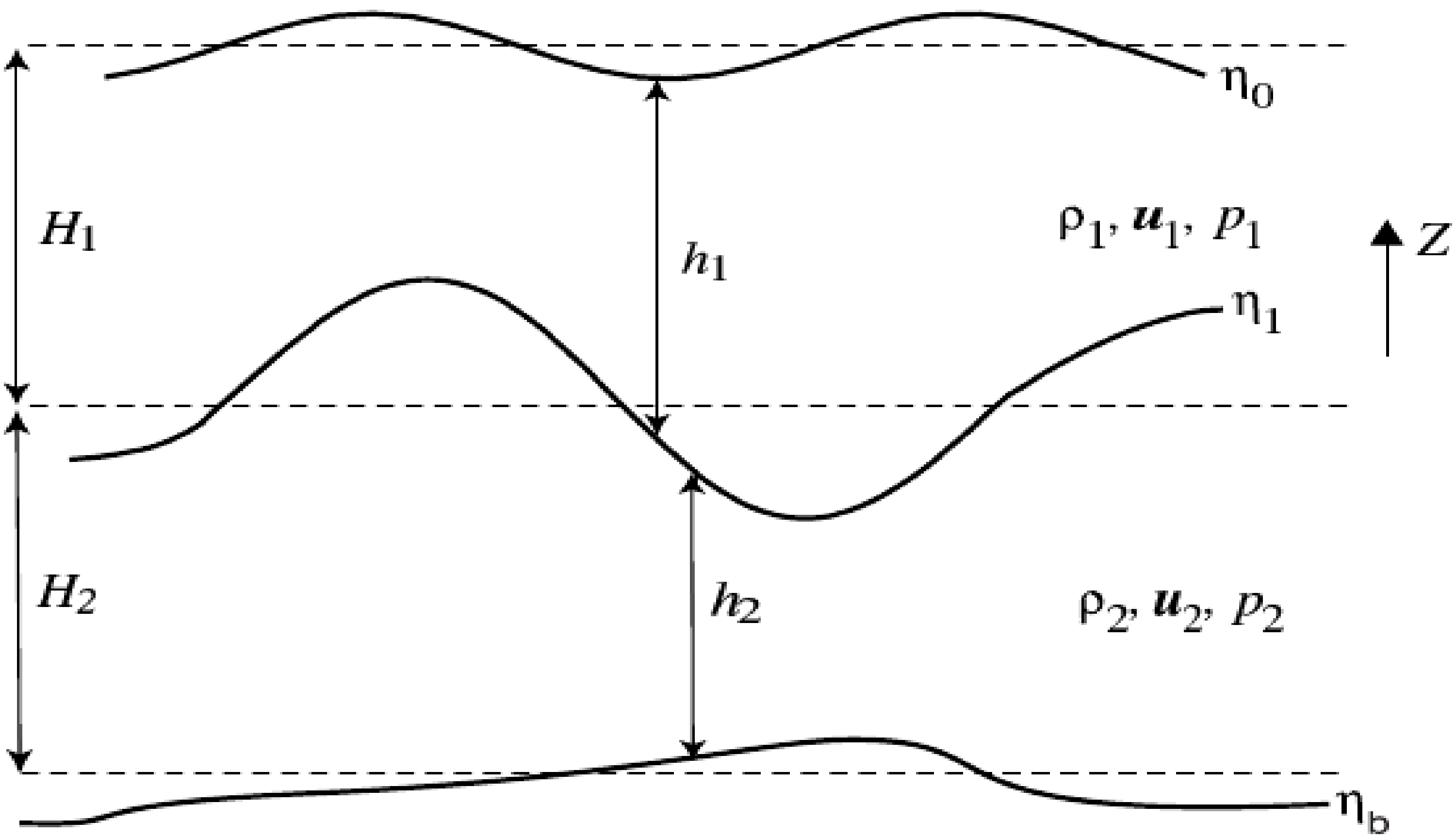
The momentum equation for each layer may then be written, in general,

$$\frac{D\boldsymbol{u}_n}{Dt} + \boldsymbol{f} \times \boldsymbol{u}_n = -\frac{1}{\rho_n}\nabla p_n,$$

Finally, the mass conservation equation for each layer has the same form as the single-layer case, and is

$$\frac{Dh_n}{Dt} + h_n \nabla \cdot \boldsymbol{u}_n = 0.$$

The **two-layer model** is the simplest model to capture the effects of stratification.



A fluid of density ρ1 lies over a denser fluid of density ρ2 . In the reduced gravity case the lower layer may be arbitrarily thick and is assumed stationary and so has no horizontal pressure gradient. In the ‘rigid-lid’ approximation the top surface displacement is neglected, but there is then a non-zero pressure gradient induced by the lid.

$$p_1 = \rho_1 g \eta_0 = \rho_1 g(h_1 + h_2 + \eta_b)$$
$$p_2 = \rho_1[g\eta_0 + g_1'\eta_1] = \rho_1\left[g(h_1 + h_2 + \eta_b) + g_1'(h_2 + \eta_b)\right].$$

The momentum equations for the two layers are then:

$$\frac{D\boldsymbol{u}_1}{Dt} + \boldsymbol{f} \times \boldsymbol{u}_1 = -g\nabla\eta_0 = -g\nabla(h_1 + h_2 + \eta_b)$$

In the top layer and

$$\begin{aligned}\frac{D\boldsymbol{u}_2}{Dt} + \boldsymbol{f} \times \boldsymbol{u}_2 &= -\frac{\rho_1}{\rho_2}\left(g\nabla\eta_0 + g_1'\nabla\eta_1\right) \\ &= -\frac{\rho_1}{\rho_2}\left[g\nabla(\eta_b + h_1 + h_2) + g_1'\nabla(h_2 + \eta_b)\right]\end{aligned}$$

In the bottom one. 2 In the Boussinesq approximation $\rho 1 = \rho 2$ is replaced by unity.

In a three layer model the dynamical pressures are found to be

$$\begin{aligned} p_1 &= \rho_1 g h \\ p_2 &= \rho_1\left[gh + g_1'(h_2 + h_3 + \eta_b)\right] \\ p_3 &= \rho_1\left[gh + g_1'(h_2 + h_3 + \eta_b) + g_2'(h_3 + \eta_b)\right], \end{aligned}$$

where $h = \eta_0 = \eta_b + h_1 + h_2 + h_3$ and $g_2' = g(\rho_3 - \rho_2)/\rho_1$.

Reduced-gravity multi-layer

is a useful model of the stratified upper ocean overlying a nearly stationary and nearly unstratified abyss. If we suppose there is a lid at the top, then the model is almost the same as previous. However, now the horizontal pressure gradient in the lowest model layer is zero, and so we may obtain the pressures in all the active layers by integrating the hydrostatic equation upwards from this layer. The dynamic pressure in the n’th layer is given by

$$p_n = -\sum_{i=n}^{i=N} \rho_1 g_i' \eta_i \qquad g_i' = g(\rho_{i+1} - \rho_i)/\rho_1.$$

Having rigid lid on the top: $$\eta_n = -\sum_{i=1}^{i=n} h_i$$

one can easily get momentum equation in each layer.

Geostrophy and thermal wind.

When the Rossby number U /f L is small the Coriolis term dominates the advective terms. In the single-layer shallow water equations:

$$\boldsymbol{f} \times \boldsymbol{u}_g = -\nabla \eta$$

and the geostrophic velocity is proportional to the slope of the surface, In both the single-layer and multi-layer case, the slope of an interfacial surface is directly related to the difference in pressure gradient on either side and so, by geostrophic balance, to the shear of the flow. This is the shallow water analog of the thermal wind relation.

Consider the interface, η, between two layers 1 and 2. The pressure in two layers is given by the hydrostatic relation and so,

$$p_1 = A(x, y) - \rho_1 g z \quad \text{(at some z in layer 1)} \tag{3.57a}$$

$$\begin{aligned} p_2 &= A(x, y) - \rho_1 g \eta + \rho_2 g(\eta - z) \\ &= A(x, y) + \rho_1 g_1' \eta - \rho_2 g z \quad \text{(at some z in layer 2)} \end{aligned} \tag{3.57b}$$

where $A(x, y)$ is the pressure where $z = 0$, but we don't need to specifiy where this is. Thus we find

$$\frac{1}{\rho_1}\nabla(p_1 - p_2) = -g_1' \nabla \eta. \tag{3.58}$$

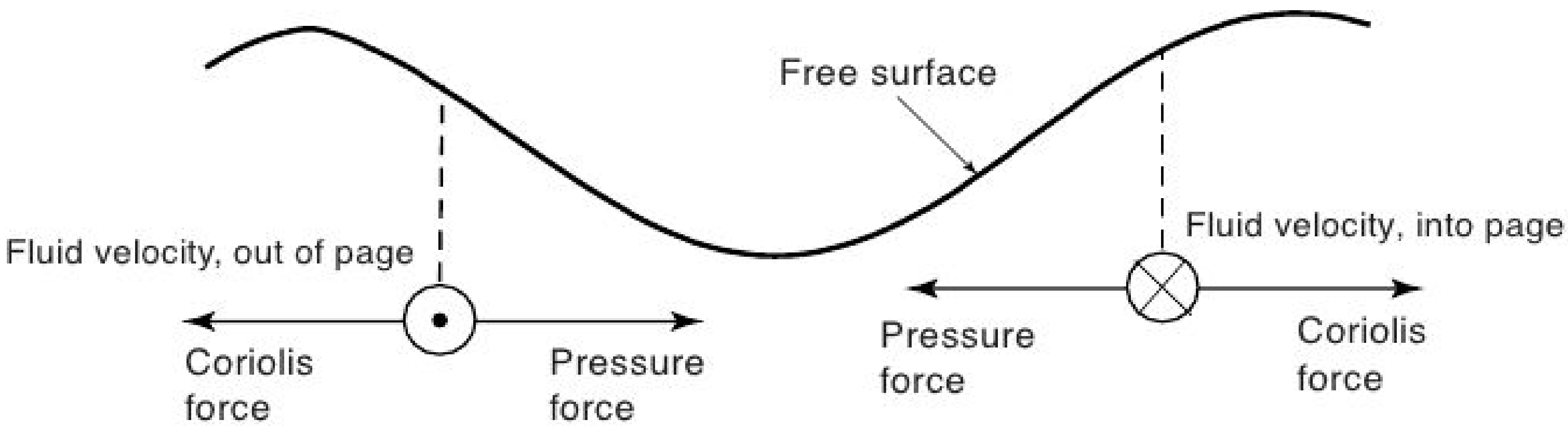


Fig. 3.6 Geostrophic flow in a shallow water system, with a positive value of the Coriolis parameter f, as in the Northern hemisphere. The pressure force is directed down the gradient of the height field, and this can be balanced by the Coriolis force if the fluid velocity is at right angles to it. If f were negative, the geostrophic flow would be reversed.

If the flow is geostrophically balanced and Boussinesq then, in each layer, the velocity obeys

$$f\boldsymbol{u}_i = \frac{1}{\rho_1}\mathbf{k}\times\nabla p_i. \tag{3.59}$$

Using (3.58) then gives

$$f(\boldsymbol{u}_1 - \boldsymbol{u}_2) = -\mathbf{k}\times g_1'\nabla\eta, \tag{3.60}$$

or in general

$$f(\boldsymbol{u}_n - \boldsymbol{u}_{n+1}) = \mathbf{k}\times g_n'\nabla\eta. \tag{3.61}$$

This is the thermal wind equation for the shallow water system. It implies the shear is proportional to the interface slope,

Imagine the atmosphere as two layers of fluid with a meridionally decreasing temperature represented by an interface that slopes upward toward the pole $\partial\eta/\partial y > 0.$
In the Northern hemisphere f is positive and we have:

$$u_1 - u_2 = \frac{g_1}{f}\frac{\partial\eta}{\partial y} > 0,$$

Indicating that such temperature gradient is associated with a positive shear.

FORM DRAG

When the interface between two layers varies with position the layers exert a pressure force on each other. If the bottom is not flat then the topography and the bottom layer can exert forces on each other. This is known as form drag, influencing momentum of the flow.

Consider a layer confined between two interfaces, η1(x,y) and η2(x,y). Over some zonal interval L the average zonal pressure force on fluid is:

$$F_p = -\frac{1}{L}\int_{x_1}^{x_2}\int_{\eta_2}^{\eta_1}\frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}z.$$

$$\begin{aligned} F_p &= -\frac{1}{L}\int_{x_1}^{x_2}\left[\frac{\partial p}{\partial x}z\right]_{\eta_2}^{\eta_1}\,\mathrm{d}x \\ &= \overline{-\eta_1\frac{\partial p_1}{\partial x}} + \overline{\eta_2\frac{\partial p_2}{\partial x}} = \overline{+p_1\frac{\partial \eta_1}{\partial x}} - \overline{p_2\frac{\partial \eta_2}{\partial x}}, \end{aligned}$$

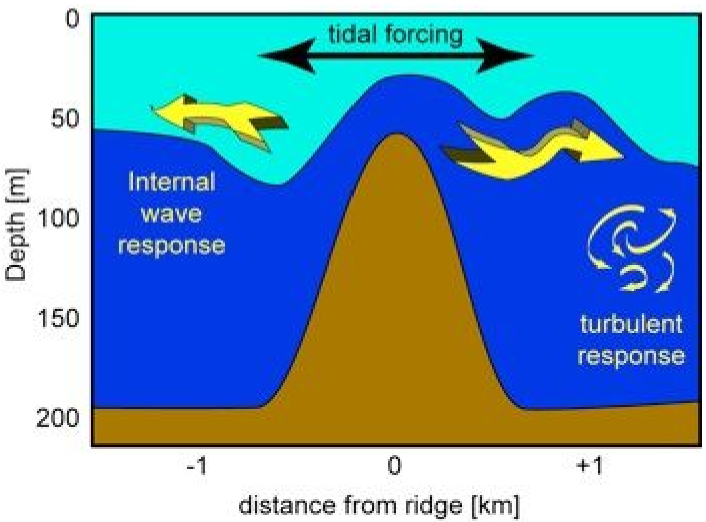
To obtain the second line we suppose that the integral is around a closed path, such as a circle of latitude, and the average is denoted with an overbar.

These terms represent the transfer of momentum from one layer to the next, and at a particular interface, i, we may define the form drag:

$$\tau_i \equiv \overline{p_i \frac{\partial \eta_i}{\partial x}} = -\overline{\eta_i \frac{\partial p_i}{\partial x}}.$$

The form drag is a stress, and as the layer depth shrinks to zero its vertical derivative is the force on the fluid.

It is a mechanism for the vertical transfer of momentum .



Three Tree Point Form Drag Experiment

The purpose of the Three Tree Point Experiment is to measure the pressure drop over a topographic feature caused by the currents flowing over top of it. We have developed special sensors to measure the force that a ridge can exert on the overlying flow, known as “form drag.” Our objective is to relate this force to other variables we can easily measure, such as the tidal strength and the density structure in Puget Sound. Three Tree Point represents an ideal natural geophysical laboratory for us to conduct these important experiments because the tidal currents are predictable and deviations from them can be associated with form drag.

http://mixing.coas.oregonstate.edu/threetree/Three_Tree_Point/Overview.html

CONSERVATION PROPERTIES OF SHALLOW WATER SYSTEMS

A material invariant: potential vorticity

The vorticity of a fluid is the curl of the velocity field:

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$$

Define shallow water vorticity as the curl of the horizontal velocity

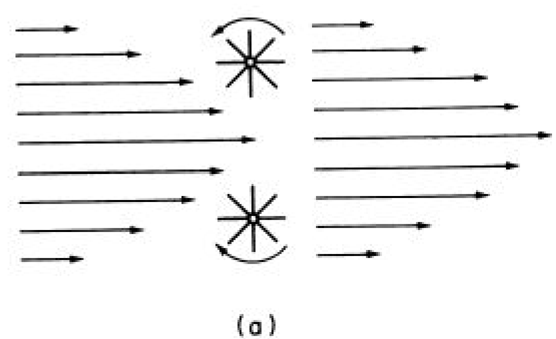
$$\partial u / \partial z = \partial v / \partial z = 0$$

$$\boldsymbol{\omega}^* = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \equiv \mathbf{k}\,\zeta.$$

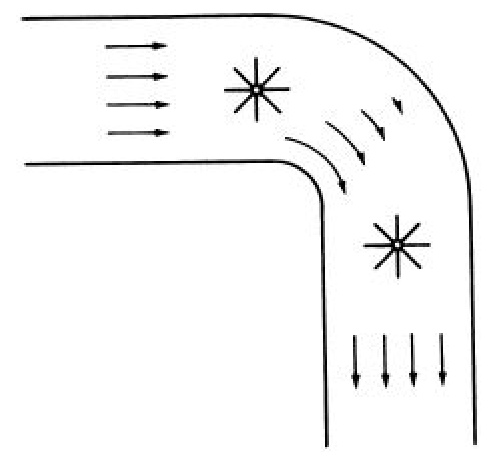
Two types of two-dimensional flow:

(a) linear shear flow with vorticity

(b) curved flow with zero vorticity.



(a)



(b)

Using the vector identity

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

we write the momentum equation $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta$ as:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega}^* \times \boldsymbol{u} = -\nabla(gh + \frac{1}{2}\boldsymbol{u}^2).$$

To obtain an evolution equation for the vorticity we take the curl of this momentum equation and use vector identity:

$$\nabla \times (\boldsymbol{\omega}^* \times \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}^* - (\boldsymbol{\omega}^* \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}^*\nabla \cdot \boldsymbol{u} - \boldsymbol{u}\nabla \cdot \boldsymbol{\omega}^*$$
$$= (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}^* + \boldsymbol{\omega}^*\nabla \cdot \boldsymbol{u},$$

Obtaining

$$\frac{\partial \boldsymbol{\omega}^*}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = -\boldsymbol{\omega}^*\nabla \cdot \boldsymbol{u},$$

Knowing that $\zeta = \mathbf{k} \cdot \boldsymbol{\omega}^*$ one may write above as:

$$\frac{\partial \zeta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\zeta = -\zeta\nabla \cdot \boldsymbol{u}.$$

The mass conservation equation may be written as:

$$-\zeta\nabla\cdot\boldsymbol{u} = \frac{\zeta}{h}\frac{\mathrm{D}h}{\mathrm{D}t}.$$

Using the last form of the momentum equation and the above one gets

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \frac{\zeta}{h}\frac{\mathrm{D}h}{\mathrm{D}t},$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta}{h}\right) = 0.$$

This is the POTENTIAL VORTICITY conservation law, and ζ/h, the potential vorticity is often denoted as Q.

Because Q is conserved on parcels, then so is any function of Q; that is, $F(Q)$ is a material invariant, where F is any function. To see this algebraically, multiply (3.76) by $F'(Q)$, the derivative of F with respect to Q, giving

$$F'(Q)\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}F(Q) = 0. \qquad (3.77)$$

Since F is arbitrary there are an infinite number of Lagrangian invariants corresponding to different choices of F.

Effects of rotation

In a rotating frame of reference, the shallow water momentum equation is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -g\nabla h,$$

which may be written in a vector invariant form as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\omega}^* + \boldsymbol{f}) \times \boldsymbol{u} = -\nabla(gh + \frac{1}{2}\boldsymbol{u}^2),$$

taking the curl of this gives the vorticity equation:

$$\frac{\partial \zeta}{\partial t} + (\boldsymbol{u} \cdot \nabla)(\zeta + f) = -(f + \zeta)\nabla \cdot \boldsymbol{u}.$$

The above is simply the equation of motion for the total or absolute vorticity:

$$\boldsymbol{\omega}_a = \boldsymbol{\omega}^* + \boldsymbol{f} = (\zeta + f)\mathbf{k}.$$

Combining it with the mass conservation gives potential vorticity in rotating coordinate frame:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{h}\right) = 0 \qquad Q \equiv (\zeta + f)/h$$

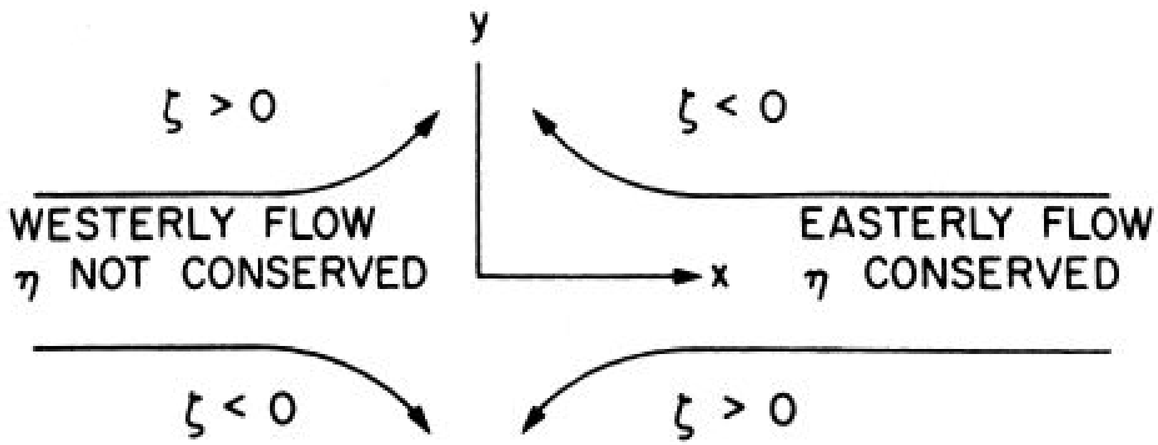


Fig. 4.8 Absolute vorticity conservation for curved flow trajectories.

Consider flow of constant depth, What are chanhes if planeray vorticity (Coriolis parameter) when changing latitude?

Westerly flows cannot turn without forcing, they are stable!

Vorticity and circulation

Vorticity itself is not a material invariant, its integral over a horizontal material area is. Consider the integral (non-rotating case):

$$C = \int_A \zeta \, \mathrm{d}A = \int_A Q h \, \mathrm{d}A,$$

Taking the material derivative of this gives

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \int_A \frac{\mathrm{D}Q}{\mathrm{D}t} h \, \mathrm{d}A + \int_A Q \frac{\mathrm{D}}{\mathrm{D}t}(h \, \mathrm{d}A).$$

The first term is zero, the second term is just the derivative of the volume of a column of fluid and it too is zero, by mass conservation. Thus,

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t} \int_A \zeta \, \mathrm{d}A = 0$$

Which means that the integral of the vorticity over a some cross-sectional area of the fluid is unchanging, although both the vorticity and area of the fluid may individually change. Using Stokes' theorem, it may be written

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t} \oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l}$$

$$\frac{DC}{Dt} = \frac{D}{Dt}\oint \boldsymbol{u} \cdot d\boldsymbol{l}$$

The above is a Kelvin circulation theorem.

Potential vorticity in the atmosphere (from Holton's book):

The potential vorticity conservation for the adiabatic atmosphere can be written as:

$$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = \text{Const}$$

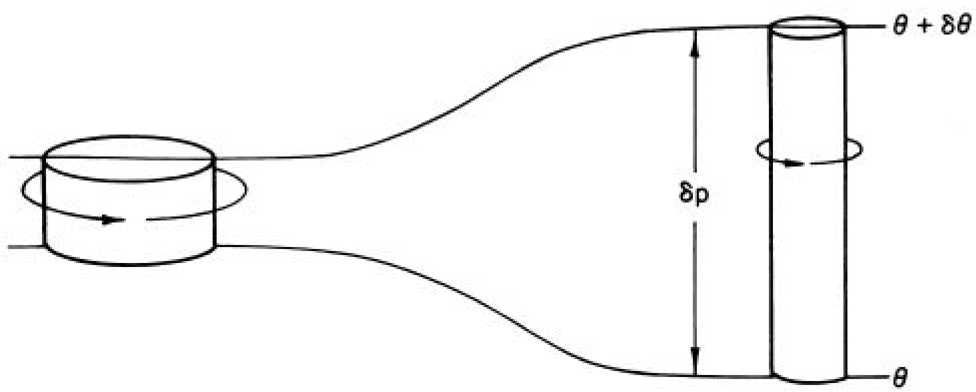
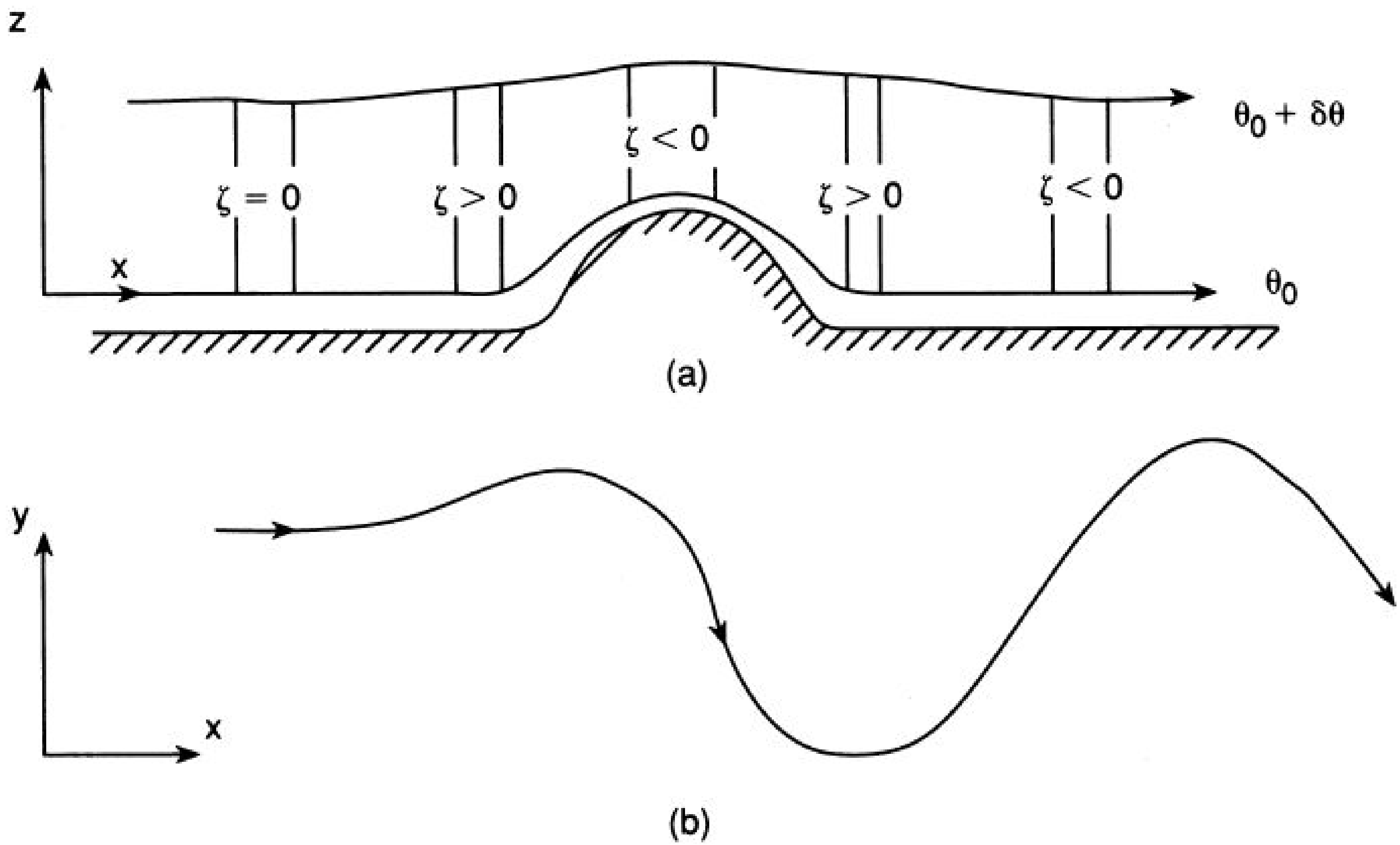


Fig. 4.7 A cylindrical column of air moving adiabatically, conserving potential vorticity.



4.9 Schematic view of westerly flow over a topographic barrier: (a) the depth of a fluid column as a function of x and (b) the trajectory of a parcel in the $(x,\ y)$ plane.

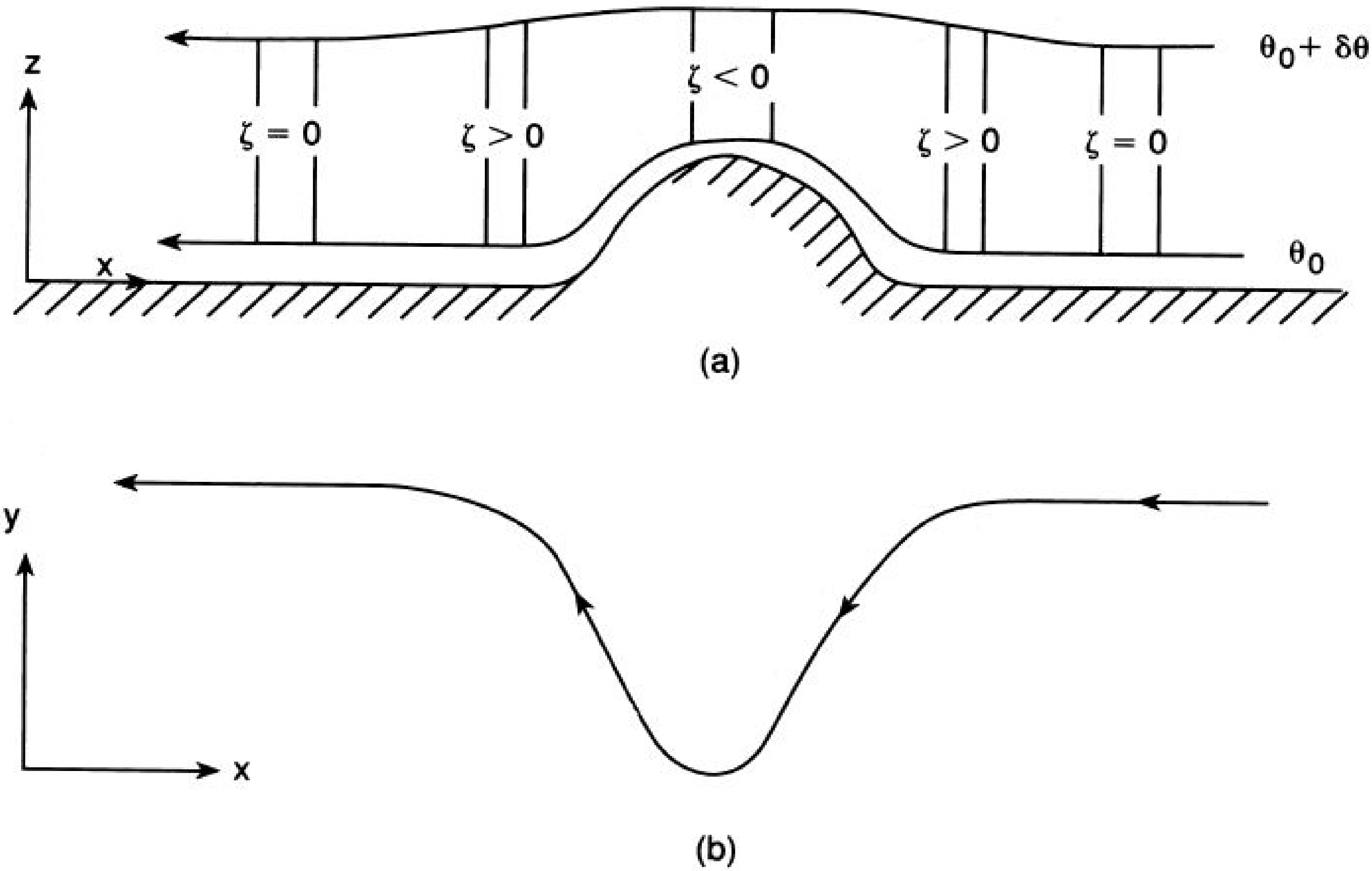


Fig. 4.10 As in Fig. 4.9, but for easterly flow.