Planetary boundary layer and atmospheric turbulence.

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Lecture 01

Figure 1: In an early study of turbulence Leonardo da Vinci wrote "Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion." (Lumley, J.L., 1997. Phys. Fluids A, 4, 203)

Turbulence, as a flow regime, can be defined by its characteristic features.

S.B. Pope Turbulent Flows (2000) „...an essential feature of turbulent flows is that the fluid velocity field varies significantly and irregularly in both position and time."

Turbulence:

Irregularity, unpredictability, unsteadiness, large variation of temporal and spatial scales which interact with each other, enhanced mixing

ENCYKLOPEDIA BRITAENNICA:

Atmospheric turbulence: small-scale, irregular air motions characterized by winds that vary in speed and direction. Turbulence is important because it mixes and churns the atmosphere and causes water vapour, smoke, and other substances, as well as energy, to become distributed both vertically and horizontally.

fractal [from the Latin word fractus ("fragmented," or "broken")], any of a class of complex geometric shapes that commonly have "fractional dimension," a concept first introduced by the mathematician Felix Hausdorff in 1918. Fractals... are capable of describing many irregularly shaped objects or spatially nonuniform phenomena in nature such as coastlines and mountain ranges…. Many fractals possess the property of self-similarity, at least approximately, if not exactly. A self-similar object is one whose component parts resemble the whole. This reiteration of details or patterns occurs at progressively smaller scales (ENCYKLOPEDIA BRITAENNICA)

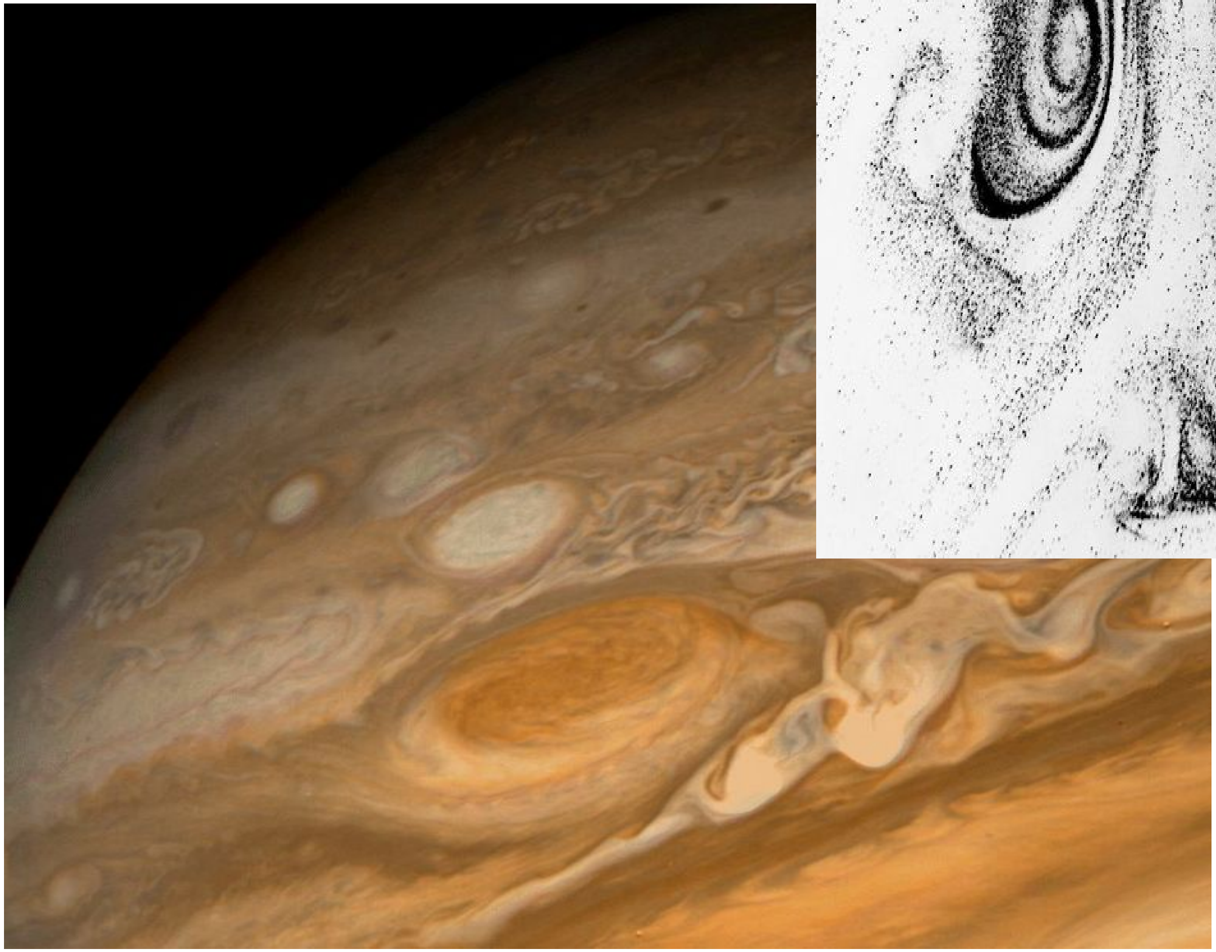
Features of turbulent flows:

Large range of temporal and spatial scales;

Nonlinear advection effects play an important role;

Unpredictable;

Irreversible.



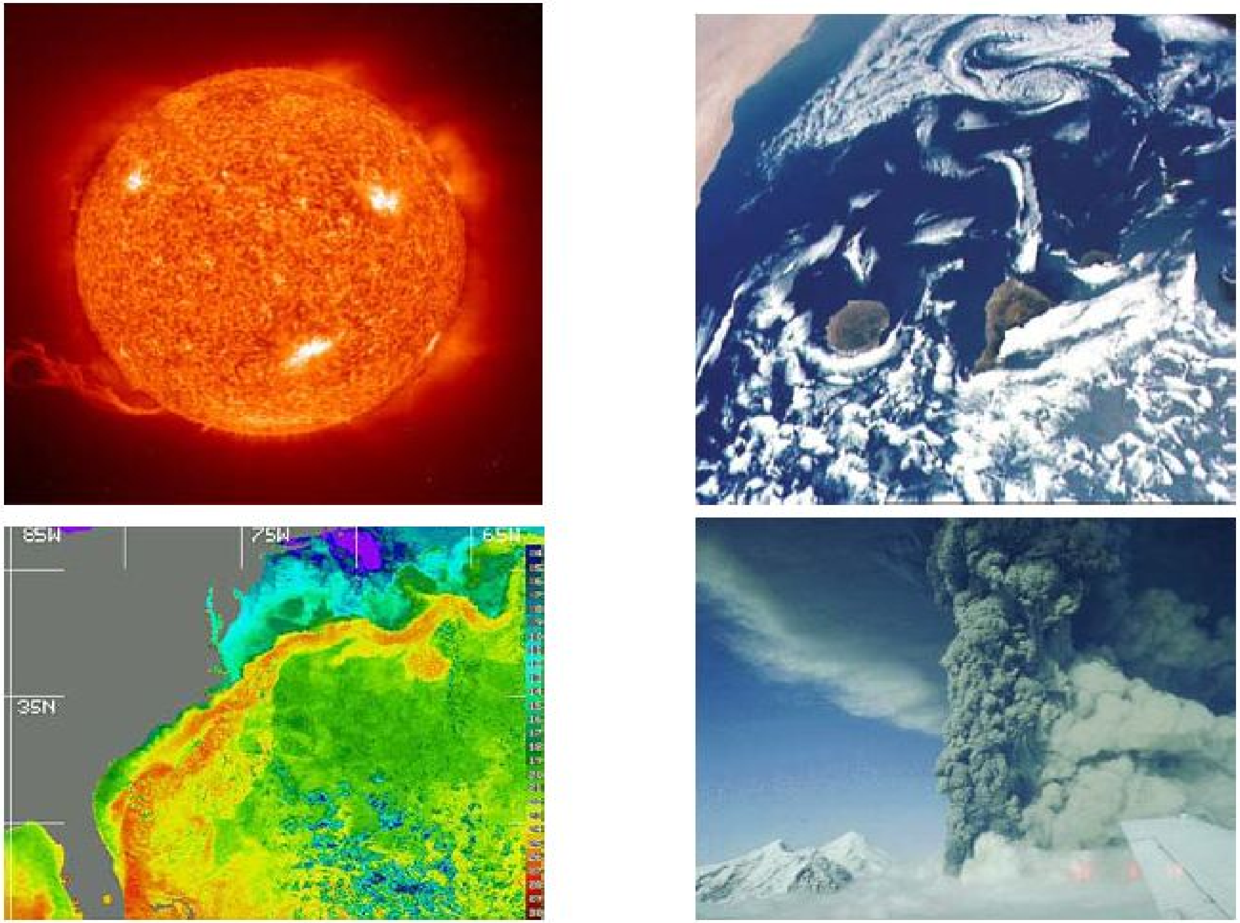


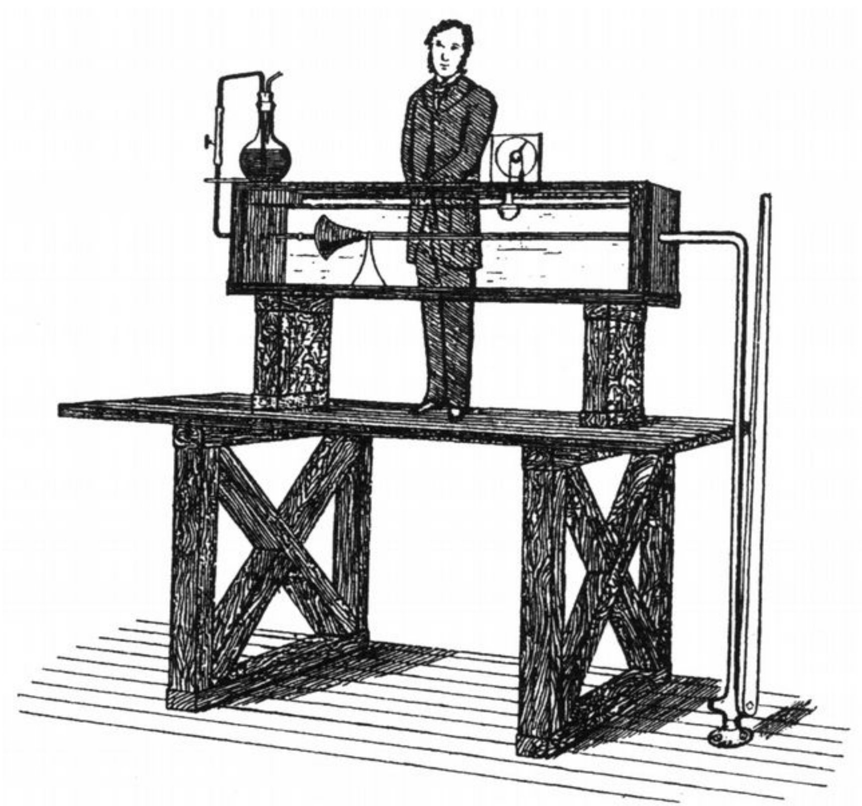
Figure 1.1: Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a vulcanic eruption.

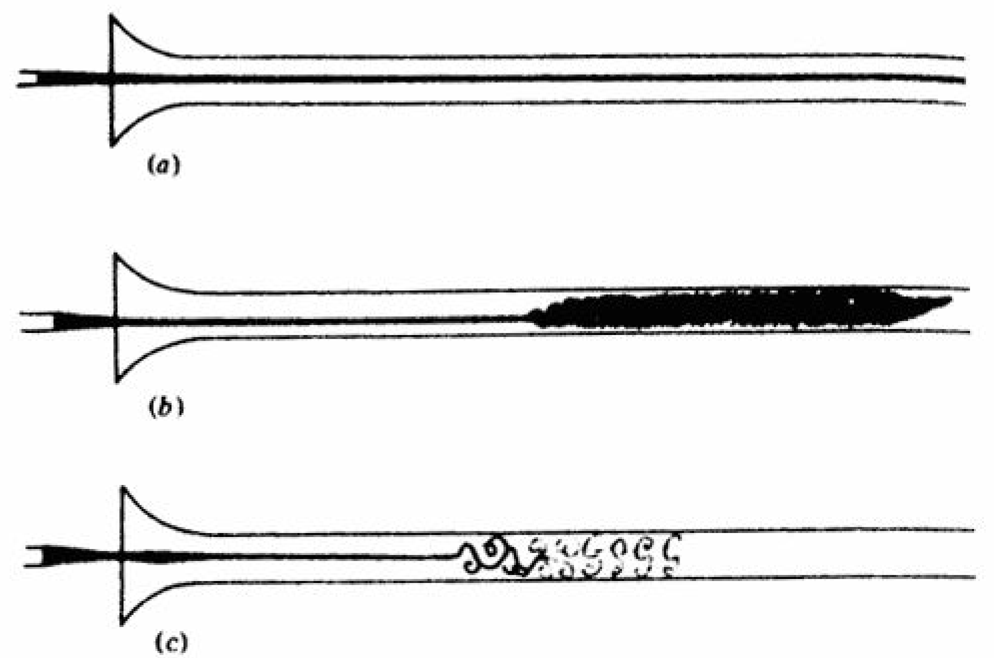
Many authors mention also:

Significant vortex streching;
Energy cascade.

Transition to turbulence

Osborne Reynolds





Osborne Reynolds:

ON THE DYNAMICAL THEORY OF INCOMPRESSIBLE VISCOUS FLUIDS AND THE DETERMINATION OF THE CRITERION.

[*From the* "Philosophical Transactions of the Royal Society," 1895.]

(*Read* May 24, 1894.)

In 1850, after Joule's discovery of the Mechanical Equivalent of Heat, Stokes showed, by transforming the equations of motion—with arbitrary stresses—so as to obtain the equations of ("Vis-viva") energy, that this equation contained a definite function, which represented the difference between the work done on the fluid by the stresses and the rate of increase of the energy, per unit of volume, which function, he concluded, must, according to Joule, represent the Vis-viva converted into heat.

This conclusion was obtained from the equations irrespective of any particular relation between the stresses and the rates of distortion. Sir G. Stokes, however, translated the function into an expression in terms of the rates of distortion, which expression has since been named by Lord Rayleigh the *Dissipation-Function.*

2. In 1883 I succeeded in proving, by means of experiments with colour bands—the results of which were communicated to the Society*—that when water is caused by pressure to flow through a uniform smooth pipe, the motion of the water is *direct*, *i.e.*, parallel to the sides of the pipe, or *sinuous*, *i.e.*, crossing and re-crossing the pipe, according as U_m, the mean velocity of the water, as measured by dividing Q, the discharge, by Δ, the area of the section of the pipe, is below or above a certain value given by

$$K\mu/D\rho,$$

where D is the diameter of the pipe, ρ the density of the water, and K a numerical constant, the value of which according to my experiments, and, as I was able to show, to all the experiments by Poiseuille and Darcy, is for pipes of circular section between

$$1900 \text{ and } 2000,$$

or, in other words, steady direct motion in round tubes is stable or unstable according as

$$\rho \frac{DU_m}{\mu} > 1900 \text{ or } < 2000,$$

the number K being thus a criterion of the possible maintenance of sinuous or eddying motion.

3. The experiments also showed that K was equally a criterion of the law of the resistance to be overcome—which changes from a resistance proportional to the velocity, and in exact accordance with the theoretical results obtained from the singular solution of the equation, when direct motion changes to sinuous, *i.e.*, when

$$\rho \frac{DU_m}{\mu} = K.$$

4. In the same paper I pointed out that the existence of this sudden change in the law of motion of fluids between solid surfaces when

$$DU_m = \frac{\mu}{\rho} K$$

proved the dependence of the manner of motion of the fluid on a relation between the product of the dimensions of the pipe multiplied by the velocity of the fluid, and the product of the molecular dimensions multiplied by the molecular velocities which determine the value of

$$\mu$$

for the fluid, also that the equations of motion for viscous fluid contained evidence of this relation.

Equations of motion in Boussinesq approximation:

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \underbrace{(\mathbf{u} \cdot \nabla)\mathbf{u}}_{inertia} &= -\frac{1}{\rho_0}\nabla p + \underbrace{\nu \nabla^2 \mathbf{u}}_{friction} + \underbrace{b\hat{z}}_{buoyancy} + \underbrace{f\hat{z} \times \mathbf{u}}_{Coriolis} \\
\frac{\partial b}{\partial t} + (\mathbf{u} \cdot \nabla) b &= \kappa \nabla^2 b, \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}$$

Notations:
p – pressure, $\mathbf{u}$ – velocity, t – time, ρ - density, b – buoyancy, f – Coriolis parameter, - ν viscosity, κ – thermal conductivity coefficient.

We can add equations for other scalars (e.g. concentration) which have the same form as (1.3) but are passive, i.e. they do not influence the velocity field (one-way coupling).

These equations have conservative integral invariants for energy, and all powers and other functionals of buoyancy, in the absence of friction and diffusion. For non-conservative dynamics, the energy and scalar variance satisfy the equations,

$$\frac{\partial E}{\partial t} = -\Upsilon, \qquad \frac{\partial B}{\partial t} = -\Upsilon_B, \tag{1.5}$$

where,

$$[E, B, \Upsilon, \Upsilon_B] = \int\int\int \mathrm{d}\boldsymbol{x} \left[e, b^2, \epsilon, \epsilon_b\right], \tag{1.6}$$

and,

$$e = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - bz, \qquad \epsilon = \nu\,\nabla\mathbf{u} : \nabla\mathbf{u}, \qquad \epsilon_b = \kappa\,\nabla b \cdot \nabla b. \tag{1.7}$$

In deriving (1.5), it is assumed that there are no boundary fluxes of energy or scalar variance. Thus, these integrals measures of the flow can only decrease with time through the action of molecular viscosity and diffusivity.

Every problem we will consider lies within the set of solutions of the PDE system in (1.2) through (1.4). No general solution is known, nor is any in prospect, because we do not know a mathematical methodology that seems powerful enough. However computers are giving us access to progressively better particular solutions, *i.e.* with progressively larger Re.

Non-dimensional form of momentum equations:

$$\mathbf{u} = \mathbf{u}^+ U,$$

$$\mathbf{x} = \mathbf{x}^+ L,$$

$$t = t^+ L/U$$

$$\frac{\partial \mathbf{u}^+}{\partial t^+} + (\mathbf{u}^+ \cdot \nabla^+)\mathbf{u}^+ \quad = \quad -\frac{1}{\rho_0}\nabla^+ \frac{p}{U^2} + \frac{\nu}{UL}(\nabla^+)^2 \mathbf{u}^+ + \frac{bL}{U^2}\hat{z} + \frac{fL}{U}\hat{z} \times \mathbf{u}^+$$

$$\frac{\partial \mathbf{u}^+}{\partial t^+} + (\mathbf{u}^+ \cdot \nabla^+)\mathbf{u}^+ \quad = \quad -\frac{1}{\rho_0}\nabla^+ \frac{p}{U^2} + \frac{1}{Re}(\nabla^+)^2 \mathbf{u}^+ + \frac{Ra}{RePe}\hat{z} + \frac{1}{Ro}\hat{z} \times \mathbf{u}^+$$

- **Inertia and friction**

The **Reynolds number** is the ratio of inertia and friction,

$$Re \equiv \frac{UL}{\nu} \tag{1.8}$$

Here U is a characteristic velocity scale, L is a length scale, and ν is the kinematic viscosity of the fluid. In turbulent flows $Re \gg 1$, advective dominance $\Rightarrow$ nonlinear dynamics $\Rightarrow$ chaotic evolution and broadband spectrum.

The focus of this course is on turbulence in the Earth's ocean and atmosphere. Typical values for ν near the Earth's surface are 1.5×10^{-5} m^2 s^{-1} for air and 1.0×10^{-6} m^2 s^{-1} for water. These values are small enough, given typical velocities U, that $Re \gg 1$ on all spatial scales L from the microscale of about 1 mm to the planetary scale of about 10^4 km. For example, $U = 1$ m s^{-1} and $L = 10^3$ m give $Re = 10^9 - 10^{10}$.

For $Re \gg 1$, the frictional term is small, at least in some sense. Paradoxically, however, the dissipation terms in (1.5) are usually not small. Thus, there must be a profound difference in solutions between the asymptotic tendency as $Re \to \infty$, and the Euler limit, $Re = \infty$ or $\nu = 0$.

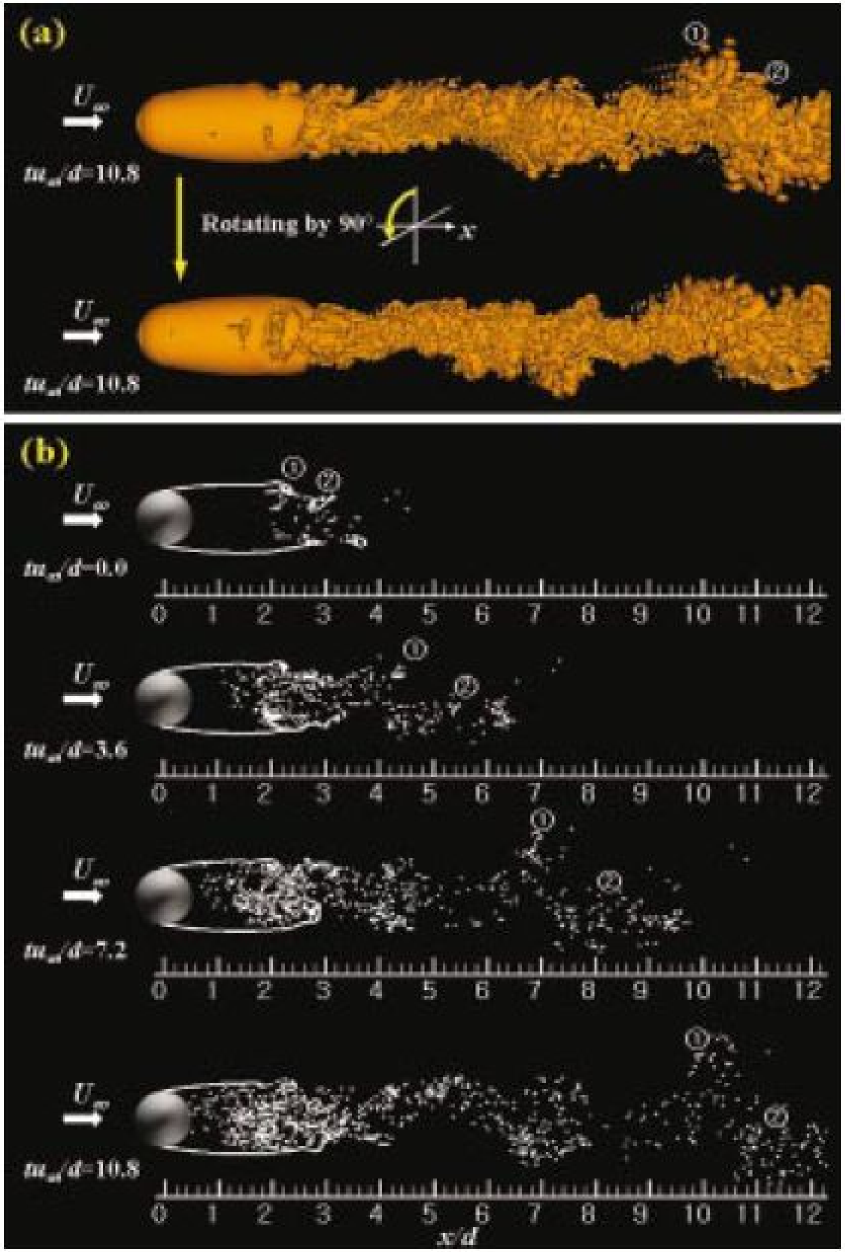


FIG. 3. Re=3700.

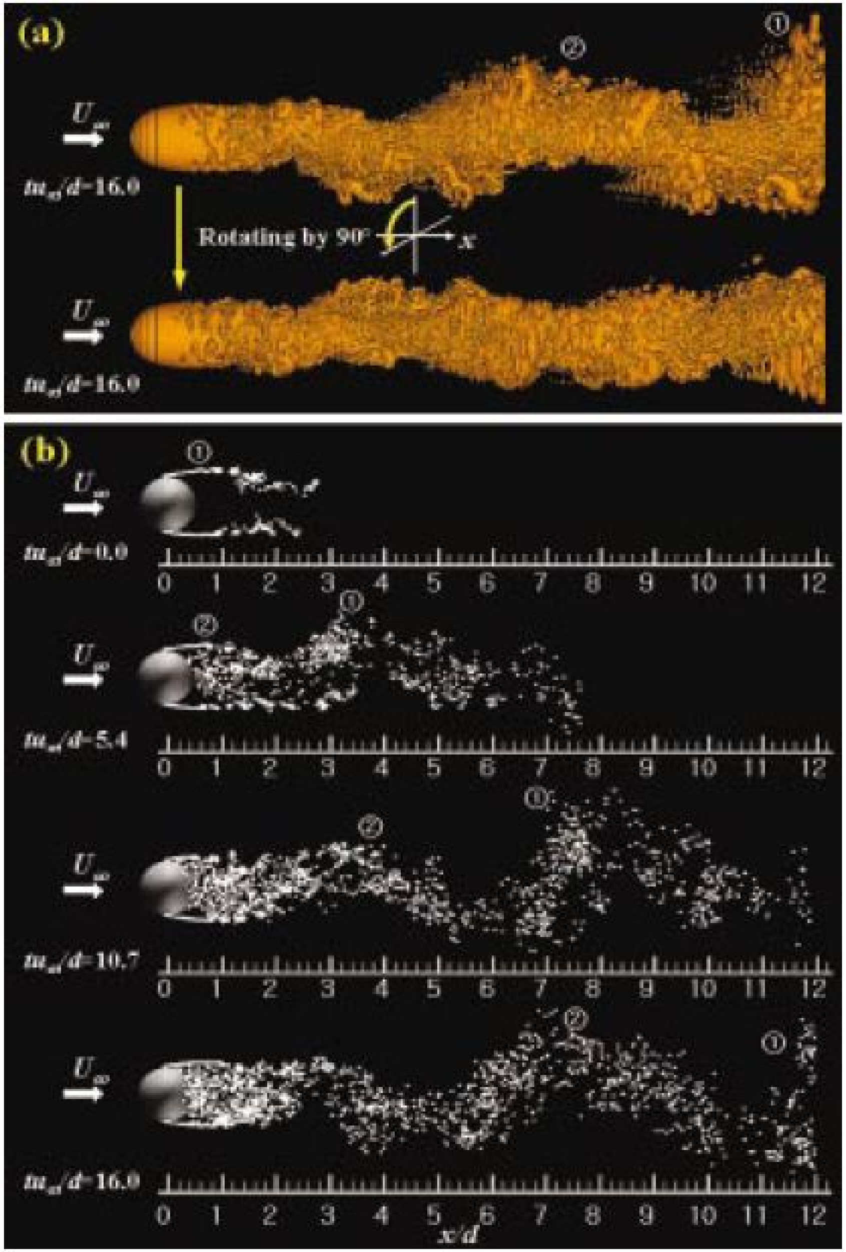


FIG. 4. Re=10^4.

$$R_e = \frac{UL}{\nu}$$

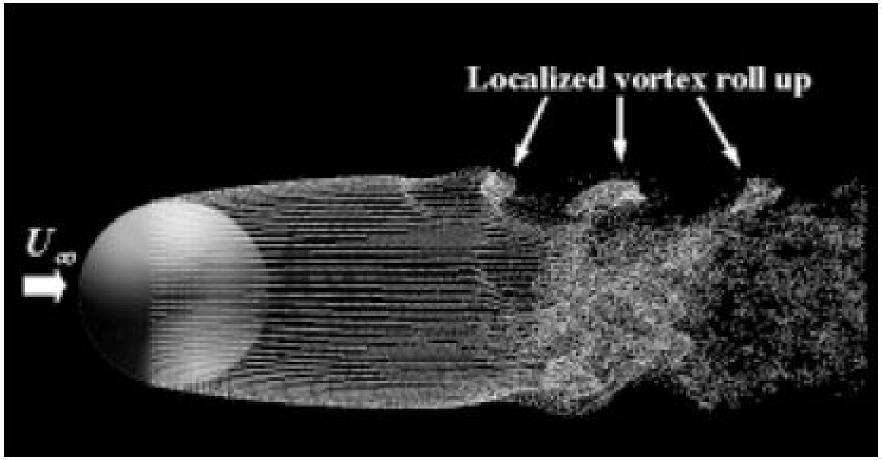


FIG. 1. Re=3700.

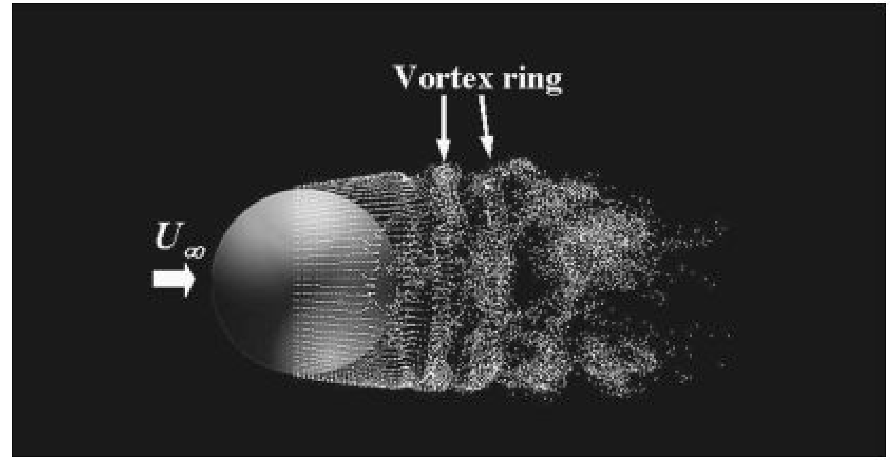
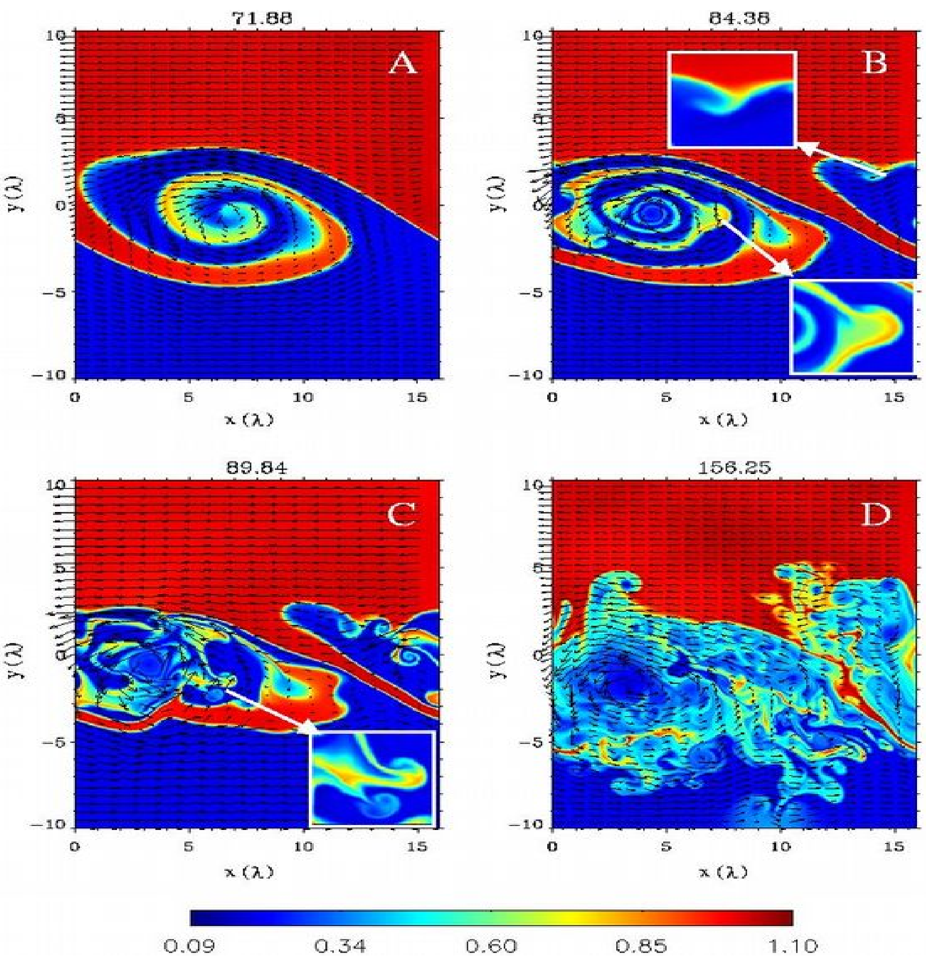
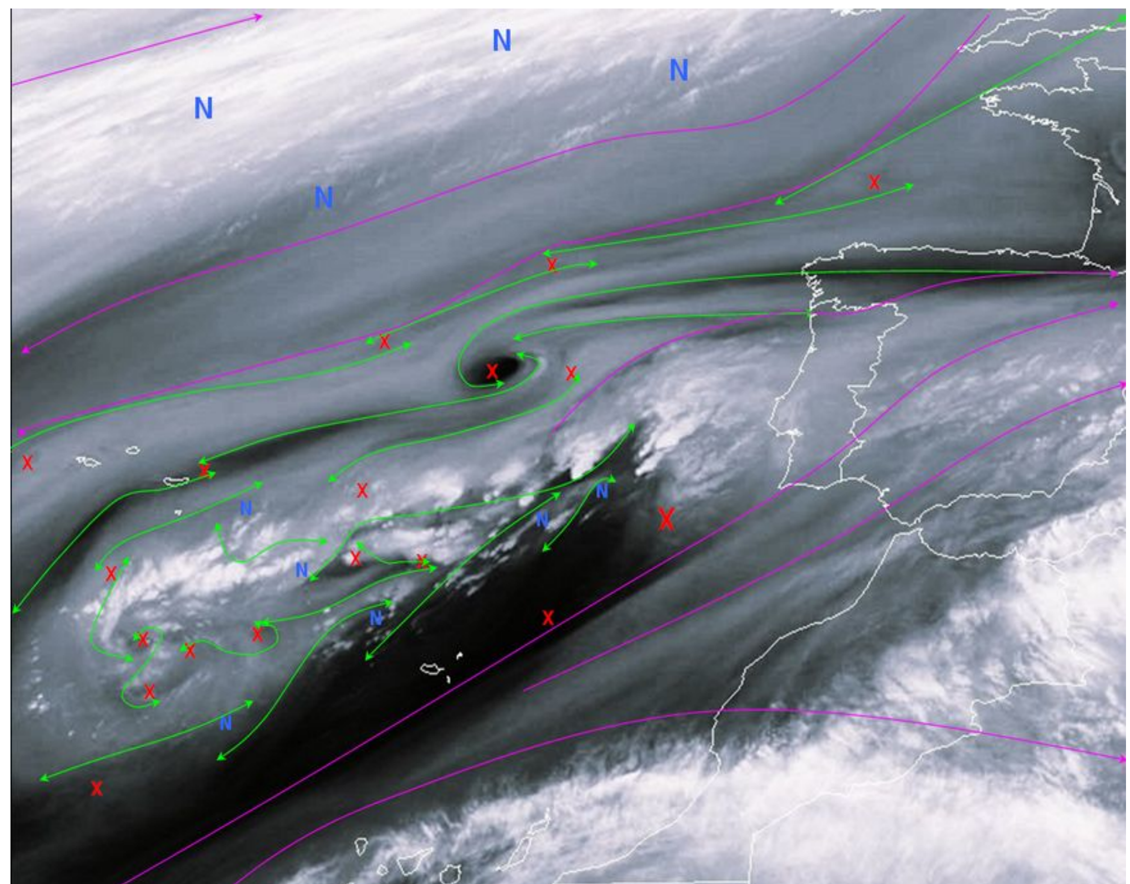


FIG. 2. Re=10^4.





- **Advection and diffusion**

$$Pe \equiv \frac{UL}{\kappa} \tag{1.9}$$

The **Peclet number** is the direct analog of Re for a conserved tracer with a diffusivity κ (*i.e.* a scalar c satisfying an equation like that of buoyancy) and measures the relative importance of advection and diffusion. At large Pe, the tracer evolution is dominated by advection. Once more, the limit $Pe \to \infty$ is very different from $Pe = \infty$, because dissipation, no matter how small, eventually is responsible for removing structure from the tracer field.

- **Friction and diffusion**

$$Pr \equiv \frac{\nu}{\kappa} \tag{1.10}$$

The frictional time scale is $t_\nu = L^2/\nu$ and the diffusive timescale $t_\kappa = L^2/\kappa$. The **Prandtl number** is defined as the ratio of these two timescales, $Pr \equiv t_\kappa/t_\nu$. The Prandtl number is a property of the fluid , not of the particular flow. Hence there is a restriction on the transfer of information from experiments with one fluid to those with another. For $Pr > 1$ the scales at which friction becomes important are larger than those for diffusion and, at some small scale, we expect to find smooth velocity fields together with convoluted tracer fields. The Prandtl numbers for air and water are 0.7 and 12.2 respectively.

- **Inertia and Coriolis**

$$Ro \equiv \frac{U}{fL} \tag{1.11}$$

The **Rossby number** Ro measures the relative importance of the real inertial forces and the fictitious Coriolis force, that appear because of the rotating reference system. Thus Ro measures the importance of rotation in the problem at hand. $Ro \gg 1$ characterizes essentially non-rotating turbulence, while $Ro \leq 1$ flows are strongly affected by rotation.

- **Buoyancy and diffusion**

$$Ra \equiv \frac{\Delta b L^3}{\kappa \nu} \tag{1.12}$$

In convective problems, motions are generated by imposing an unstable density stratification on the fluid ($\partial b/\partial z < 0$). In these problems, it is useful to characterize turbulence in terms of the **Rayleigh number**, *i.e.* the ratio of the diffusive $t_\kappa = L^2/\kappa$ and buoyancy $t_b = (L/\Delta b)^{1/2}$ timescales. The buoyancy scale Δb is the buoyancy difference maintained across the layer depth L through external forcing. If the forcing is imposed by maintaining a temperature difference ΔT, then one has $\Delta b = g\alpha\Delta T$, where α is the coefficient of thermal expansion of the fluid, and g the acceleration of gravity. Convection starts if $t_\kappa \gg t_b$, *i.e.* if $RaPr \gg 1$, when diffusion is too slow to change substantially the buoyancy of water/air parcels as they rise.

- **Buoyancy and inertia**

$$Ri \equiv \frac{\partial b/\partial z}{|\partial \mathbf{u}/\partial z|^2} \tag{1.13}$$

In the presence of stable buoyancy stratification, vertical motions tend to be suppressed, but turbulence can still emerge, if there is enough energy in the horizontal velocity field. A useful parameter to characterize the flow in these problems is the ratio of the buoyancy timescale $t_b = (L/\Delta b)^{1/2} = 1/(\partial b/\partial z)^{1/2}$ and the inertial timescale due to horizontal shears in the flow $t_i = L/U = 1/(\partial \mathbf{u}/\partial z)$. This ratio is called the gradient Richardson number Ri. If $Ri \ll 1$, buoyancy can be neglected in the momentum equations, and it becomes a passive scalar with no feedbacks on the dynamics.

A final remark about the only term that never appeared explicitly in the nondimensional numbers presented: the pressure force. Pressure can be formally eliminated from the equations. This is a consequence of the Boussinesq approximation. We simply need to take the divergence of the momentum equation in (1.2) and note that $\nabla \cdot \mathbf{u}_t = 0$ because of incompressibility. This yields the relation,

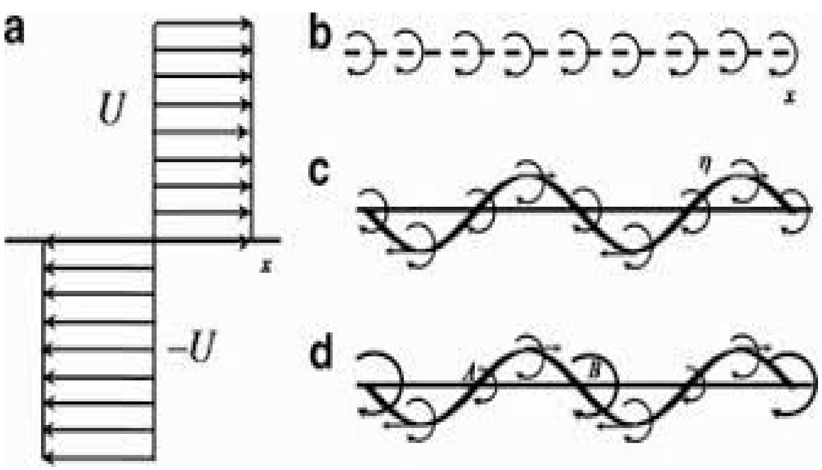
$$\nabla^2 p = \rho_0 \nabla \cdot \left[-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + b\hat{z} - f\hat{z} \times \mathbf{u} \right]. \tag{1.14}$$

Since there are no time derivatives in (1.14), pressure is a purely diagnostic field, which is wholly slaved to $\mathbf{u}$. It can be calculated from (1.14) and then substituted for the pressure gradient force in the momentum equations. Its role is to maintain incompressibility under the action of all other forces. Therefore it would be redundant to introduce nondimensional parameters involving pressure, because those parameters could be expressed as combinations of the other parameters already discussed.

Kelvin-Helmholz instability

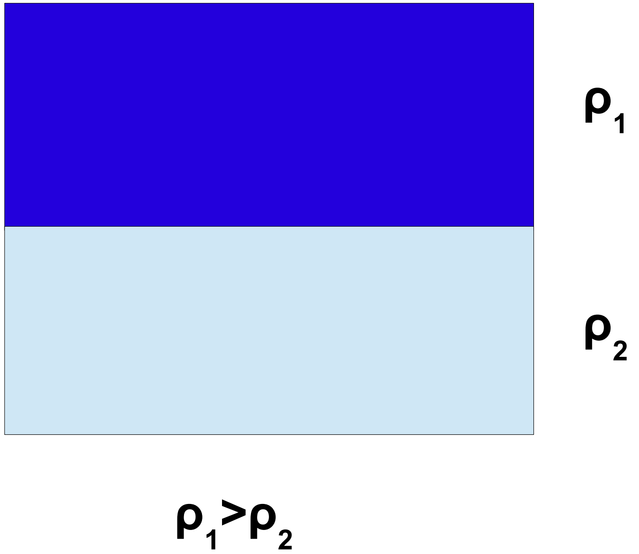
Instability of the interface between two fluids of different densities and different speeds. Example is wind blowing over water: The instability manifests in waves on the water surface.

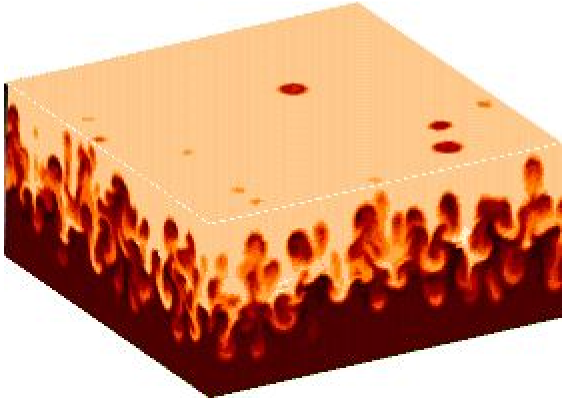
In the absense of surface tension, the instability develops for all speeds (flow is unconditionally unstable).

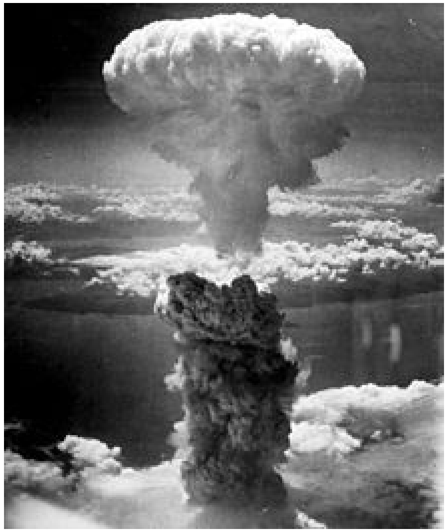


Rayleigh-Taylor instability

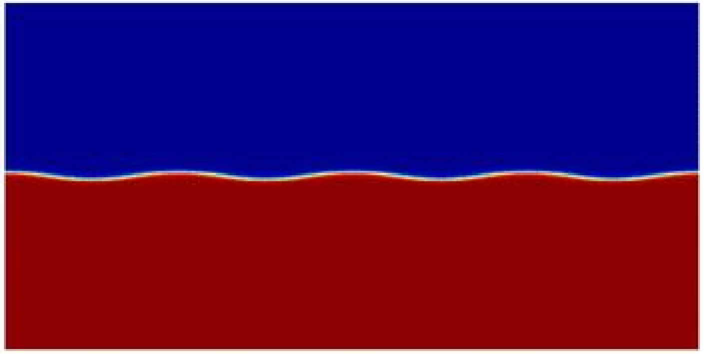
Instability of the interface between two fluids, the heaviest above the lightest. In this configuration surface tension plays a stabilizing role while gravity is destabilizing.



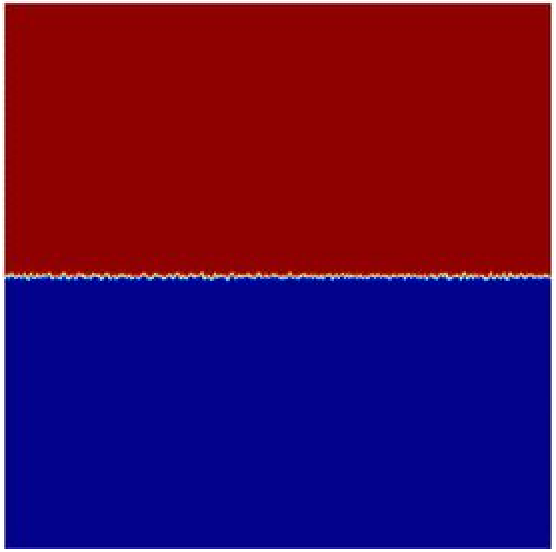




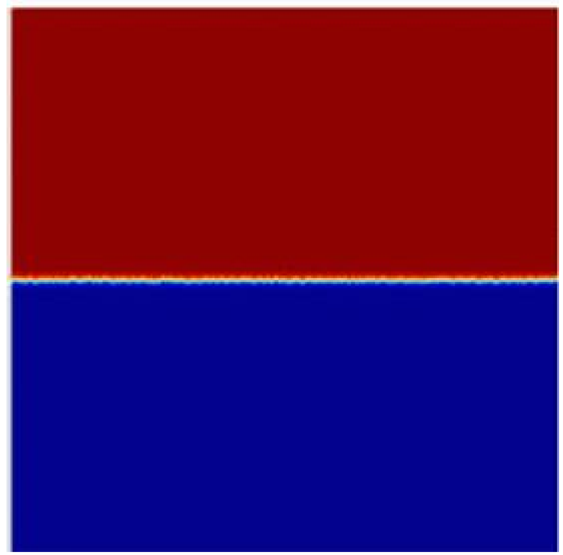
Pictures from: http://hmf.enseeiht.fr/travaux/CD0001/travaux/optmfn/hi/01pa/hyb72/rt/rt.htm
https://en.wikipedia.org/wiki/Mushroom_cloud



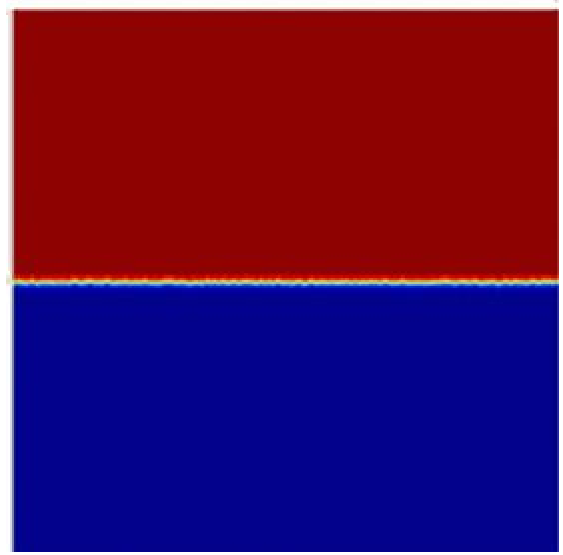
Kelvin-Helmholtz instability with Ri=.038, Re=5000



Rayleigh-Taylor instability with Ri=infinity,
Ra=6000000, Re=300



Dominant shear instability with Ri=-0.038,
Ra=3400000, Re=3600



Dominant convective instability with Ri=-1.34, Ra=31000000, Re=1800

How to find solutions of the equations of motion or determine charactieristic features of these equations?

- exact analytical solutions for certian flow cases and certian parameters
-analytical approximate solutions
-numerical solutions: DNS
-statistical approaches which determine characteristic features of equations
-dimensional analysis

Deriviation of RANS Equations

The basic tool required for the derivation of the RANS equations from the instantaneous Navier-Stokes equations is the *Reynolds decomposition*. Reynolds decomposition refers to separation of the flow variable (like velocity u) into the mean (time-averaged) component ($\bar{u}$) and the fluctuating component (u').[2] Thus,

$$u(\mathbf{x},t) = \bar{u}(\mathbf{x}) + u'(\mathbf{x},t)$$ [3]

where, $\mathbf{x} = (x,y,z)$ is the position vector.

The following rules will be useful while deriving the RANS. If f,and$g$ are two flow variables (like density (ρ), velocity (u), pressure (p), etc.) and s is one of the independent variables (x,y,z, or t) then,

$$\bar{\bar{f}} = \bar{f}$$

$$\overline{f+g} = \bar{f} + \bar{g}$$

$$\overline{\bar{f}g} = \bar{f}\bar{g}$$

$$\overline{fg} \neq \bar{f}\bar{g}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}$$

Now the Navier-Stokes equations of motion [4] for an incompressible Newtonian fluid are:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Subsituting, $u_i = \bar{u}_i + u_i', p = \bar{p} + p'$, etc. [5] and taking a time-average of these equations yeilds,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u_j' \frac{\partial u_i'}{\partial x_j}} = \bar{f}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

The momentum equation can also be written as, [6]

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} = \bar{f}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

On further manipulations this yields,

$$\rho\frac{\partial \bar{u}_i}{\partial t} + \rho\frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} = \rho\bar{f}_i + \frac{\partial}{\partial x_j}\left[-\bar{p}\delta_{ij} + 2\mu\bar{S}_{ij} - \rho\overline{u_i' u_j'}\right]$$

where, $\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$ is the mean rate of strain of strain tensor.

Notes

1. ^ The true time average ($\bar{X}$) of a variable (x) is defined by,

$$\bar{X} = \lim_{T\rightarrow\infty} \frac{1}{T} \int_{t_0}^{t_0+T} x\, dt$$

In general for a time-average to be useful quantity, it is required that the average ($\bar{X}$) be independent of the starting time (t_0). This constraint is important for otherwise using time-averaging would be meaning less. This implies that the average value ($\bar{X}$) is independent of time (t). Since it is not possible to integrate over an infinte time period, it is necessary to restict the integration to some finite, yet large time interval. This interval is so selected that the term $\bar{X}$ is independent of the length of the interval (T). However, the independence from t_0 can no longer be ensured. Only in case of steady flows will $\bar{X}$ be independent of both t_0 and T. Thus,

$$\bar{X}(t_0) = \frac{1}{T} \int_{t_0}^{t_0+T} x\, dt$$

2. ^ By definition, the mean of the fluctuating quantity is zero($\overline{u'} = 0$).
3. ^ Some authors prefer using U instead of $\bar{u}$ for the mean term (since an overbar is used to represent a vector). Also it is common practice to represent the fluctuating term u' by u, even though u refers to the instantaneous value. This is possible because the two terms do not appear simultaneously in the same equation. to avoid confusion we will use $u, \bar{u},$ and u' to represent the instantaneous, mean and fluctuating term.
4. ^ The equations are expressed in tensor notation, which greatly simplifies the maths.

5. ^

$$\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial x_i}=0$$

$$\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial t}+\left(\bar{u_j}+u_j'\right)\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial x_j}=\left(\bar{f_i}+f_i'\right)-\frac{1}{\rho}\frac{\partial\left(\bar{p}+p'\right)}{\partial x_i}+\nu\frac{\partial^2\left(\bar{u_i}+u_i'\right)}{\partial x_j\partial x_j}$$

Time averaging these equations yeilds,

$$\overline{\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial x_i}}=0$$

$$\overline{\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial t}}+\overline{\left(\bar{u_j}+u_j'\right)\frac{\partial\left(\bar{u_i}+u_i'\right)}{\partial x_j}}=\overline{\left(\bar{f_i}+f_i'\right)}-\frac{1}{\rho}\overline{\frac{\partial\left(\bar{p}+p'\right)}{\partial x_i}}+\nu\overline{\frac{\partial^2\left(\bar{u_i}+u_i'\right)}{\partial x_j\partial x_j}}$$

Note that the nonlinear terms (like $\overline{u_i u_i}$) can be simplified to,

$$\overline{u_i u_i}=\overline{\left(\bar{u_i}+u_i'\right)\left(\bar{u_i}+u_i'\right)}=\overline{\bar{u_i}\bar{u_i}+\bar{u_i}u_i'+u_i'\bar{u_i}+u_i'u_i'}=\bar{u_i}\bar{u_i}+\overline{u_i'u_i'}$$

6. ^ This follows from the mass conservation equation which gives,

$$\frac{\partial u_i}{\partial x_i}=\frac{\partial \bar{u_i}}{\partial x_i}=\frac{\partial u_i'}{\partial x_i}=0$$