Planetary boundary layer
and atmospheric turbulence.

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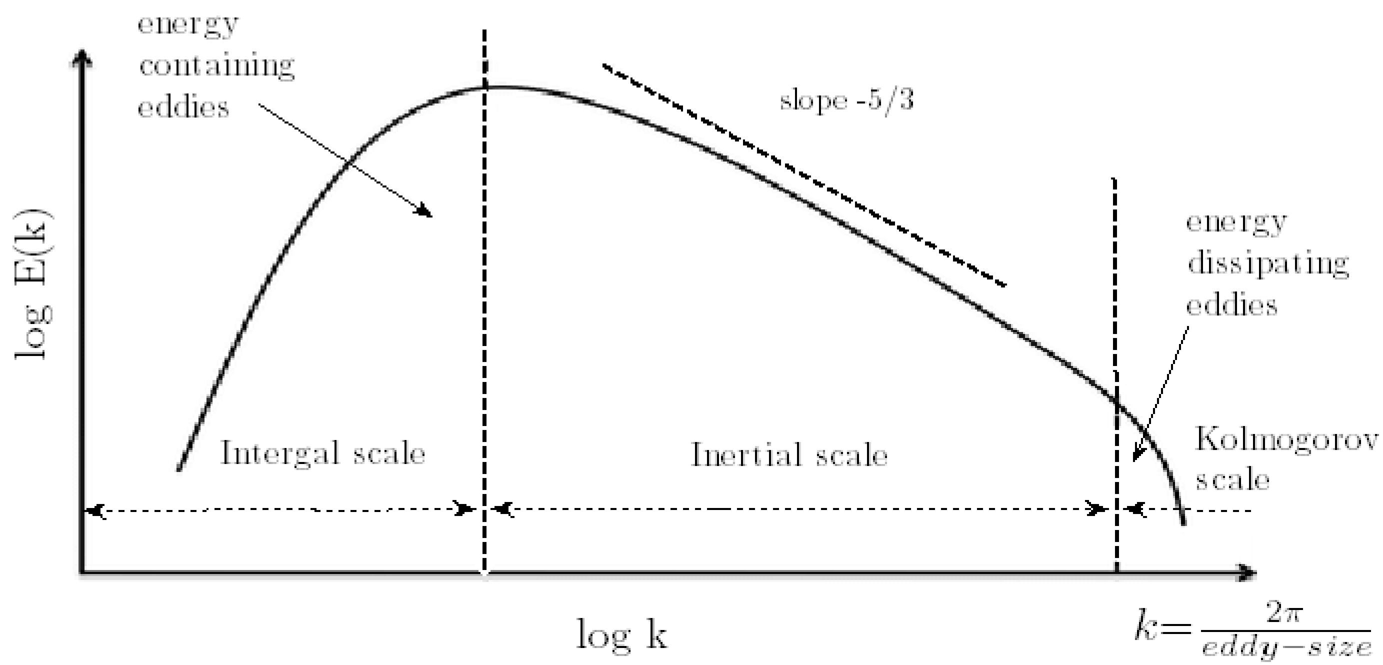
Lecture 03



# Turbulent kinetic energy - TKE.

Adding the contributions due to the 3 velocity components and rewriting in Einstein notation we have

$$\left(\frac{\partial}{\partial t}+\overline{U}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{u_i'^2}}{2}=\frac{\partial}{\partial x_j}\left(-\frac{1}{\rho_0}\overline{u_i'p'}\delta_{ij}+\nu\frac{\partial}{\partial x_j}\frac{\overline{u_i'^2}}{2}-\overline{u_j'u_i'u_i'}\right)$$
$$-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}-\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{U}_i+\overline{b'w'} \tag{3.14}$$

Hence TKE is generated by (a) shear production,

$$P=-\overline{u_j'u_i'}\frac{\partial}{\partial x_j}\overline{U}_i \tag{3.15}$$

and (b) buoyant production

$$B=\overline{b'w'} \tag{3.16}$$

F - production

and lost through dissipation

$$\epsilon=\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{3.17}$$

ε - dissipation

The buoyant production term may be either positive (generation of kinetic energy, loss of potential energy) or negative (loss of KE, increase in PE).

# Homogeneous and isotropic turbulence.

For turbulence to be isotropic: (a) Coriolis and buoyancy must be unimportant and therefore neglected (b) There must be no large-scale shear in any direction. If turbulence is isotropic then there are no spatial gradients in any averaged quantities. Hence for isotropic, homogeneous turbulence, the kinetic energy equation reduces to:

$$\frac{\partial}{\partial t}\frac{\overline{u_i'^2}}{2} = -\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{4.1}$$

or

$$\frac{d}{dt}E = -\epsilon \tag{4.2}$$

Turbulent kinetic energy $E$ is therefore a conserved quantity of the motion, known as a **quadratic invariant**. The variance of a passive tracer is another quadratic invariant: e.g.

$$\frac{d}{dt}\frac{\overline{T'^2}}{2} = -\kappa\overline{\nabla T'.\nabla T'} \tag{4.3}$$

TKE changes only by viscous dissipation. Of course this in unsustainable - a source of kinetic energy is needed. TKE sources (shear production, buoyant production) are NOT isotropic and homogeneous. We sidestep this contradiction by assuming that for large Reynolds numbers, although isotropy and homogeneity are violated by the mechanism producing the turbulence, they still hold at small scales and away from boundaries. Then the turbulence production can be represented simply by a forcing term $F$, assumed to be isotropic and homogeneous:

$$\frac{d}{dt}E = -\epsilon + F \tag{4.4}$$

Changes of TKE

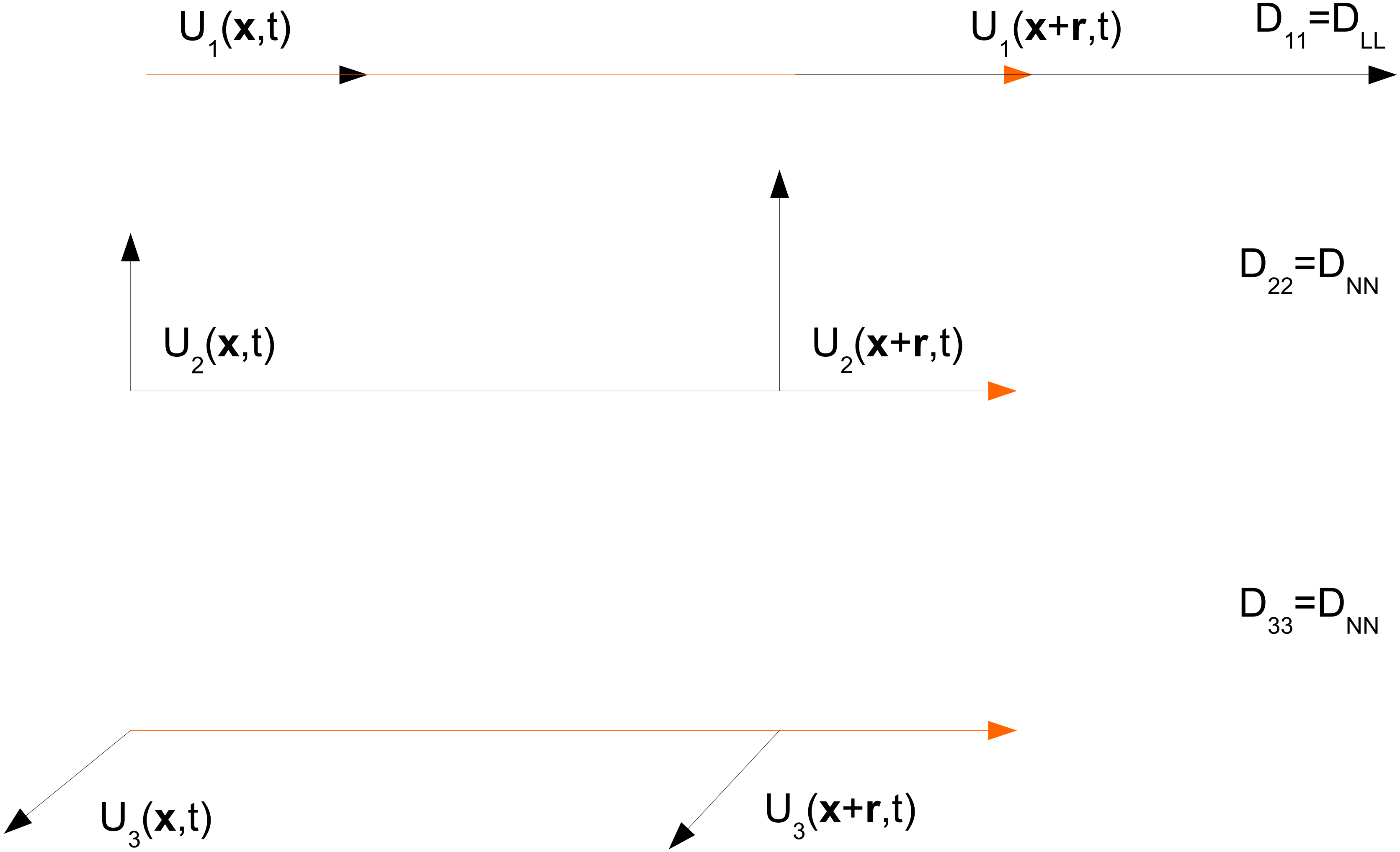
Dissipation of TKE

Production of TKE

In case of stationary turbulence production is balanced by dissipation.

Structure functions:

$$D_{ij}(\boldsymbol{x},\boldsymbol{r},t) = \langle [U_i(\boldsymbol{x}+\boldsymbol{r},t) - U_i(\boldsymbol{x},t)]\,[U_j(\boldsymbol{x}+\boldsymbol{r},t) - U_j(\boldsymbol{x},t)]\rangle$$



Structure functions in homogeneous isotropic turbulence (HIT):

$$D_{ij}(\boldsymbol{r},t) = D_{NN}(r,t)\delta_{ij} + \left[D_{LL}(r,t) - D_{NN}(r,t)\right]\frac{r_i r_j}{r^2}$$

$$\frac{\partial D_{ij}(\boldsymbol{r},t)}{\partial r_i} = 0.$$

$$D_{NN}(r,t) = D_{LL}(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} D_{LL}(r,t)$$

In HIT structure functions $D_{ij}(\mathbf{r},t)$ are determined by the single scalar function $D_{LL}(r,t)$.

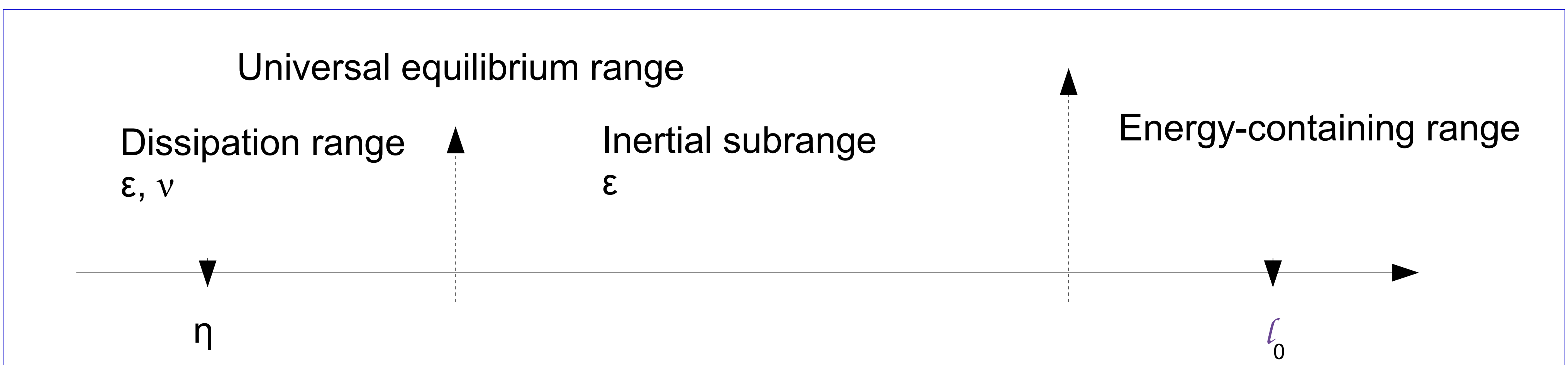
## Kolmogorovs theory (1941):

H1. Hypothesis of local isotropy – at sufficiently high Re number the small-scale motins are statistically isotropic

H2. First similarity hypothesis – in every turbulent flow at sufficiently high Re the statistics of small-scale motions have a universal form that is uniquely determined by $\nu$ and $\varepsilon$.

$$\eta=(\nu^3/\varepsilon)^{1/4} \qquad u_\eta=(\nu\varepsilon)^{1/4} \qquad \tau_\eta=(\nu/\varepsilon)^{1/2}$$

H3. Second similarity hypothesis - in every turbulent flow at sufficiently high Re the statistics of the motions of scale $\ell$ such that $\ell_0$ » $\ell$ » $\eta$ have a universal form that is uniquely determined by $\varepsilon$, independent of $\nu$.



Kolmogorovs theory (1941) [from Frisch (1990)]:

The Navier–Stokes equations for incompressible fluid flow possess a number of symmetries (invariance groups). When boundaries are ignored, the symmetries include: space and time translations, rotations, parity (space and velocity reversal) and galilean transformations. If the viscosity $\nu = 0$, an infinite class of additional symmetries appears, the scaling transformations:

$$\boldsymbol{r} \to \lambda \boldsymbol{r}, \quad \boldsymbol{v} \to \lambda^{h} \boldsymbol{v}, \quad t \to \lambda^{1-h} t, \quad \lambda \in \boldsymbol{R}_{+}. \tag{1}$$

Here, $t$, $\boldsymbol{r}$ and $\boldsymbol{v}$ are, respectively, the time, position and velocity variables. It is assumed that pressure has been eliminated from the Navier–Stokes equation through use of the incompressibility constraint. The different scaling groups are labelled by the scaling exponent $h \in \boldsymbol{R}$. (..............)

I shall present the reformulation in the form of numbered hypotheses.

H1. *In the limit of infinite Reynolds numbers, all the possible symmetries of the Navier–Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.*

The words 'small scales' can be technically defined by considering velocity increments over a distance $l$ small compared to the integral scale $l_0$:

Velocity differences at scale $l$

$$\delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{l}) = \boldsymbol{v}(\boldsymbol{r} + \boldsymbol{l}) - \boldsymbol{v}(\boldsymbol{r}). \tag{2}$$

We may then define, for example, statistical invariance under space-translations (homogeneity) by:

$$\delta \boldsymbol{v}(\boldsymbol{r} + \boldsymbol{q}, \boldsymbol{l}) \stackrel{\mathrm{L}}{=} \delta \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{l}), \quad q \ll l_0, \tag{3}$$

where $=^{\mathrm{L}}$ means 'equality in law' (identical statistical properties).

Since there is an infinity of different possible scaling exponents $h$, additional assumptions are needed.

H2. *Under the same assumptions as in H1, the turbulent flow is assumed to be self-similar at small scales, i.e. to possess a single scaling exponent* $h$.

The value of $h$ is obtained from

H3. *Under the same assumptions as in H1, the turbulent flow is assumed to have a finite non-vanishing mean rate of dissipation* $\epsilon$ *per unit mass.*†

From H2 and H3, the value of the scaling exponent can be readily obtained. Indeed, Kolmogorov (1941 *c*) has derived the following relation from the Navier–Stokes equation, under the sole assumptions of homogeneity, isotropy and finite mean energy dissipation:

3rd order structure function

$$S_3(l) \equiv \langle (\delta v_{\parallel}(\boldsymbol{r}, l))^3 \rangle = -\tfrac{4}{5}\epsilon l. \tag{4}$$

Here, $\delta v_{\parallel}$ denotes the component of the velocity increment parallel to the displacement vector $l$. The function $S_3$ is called the third order (longitudinal) structure function. The increment $l$ is assumed by Kolmogorov to be small compared to the integral scale $l_0$. With the assumption H2, under rescaling of the increment $l$ by a factor $\lambda$, the left-hand side of (4) changes by a factor $\lambda^{3h}$ while the right-hand side changes by a factor $\lambda$. Hence,

$$h = \tfrac{1}{3}. \tag{5}$$

Universal scaling exponent

Under the assumption that moments of arbitrary integer order $p$ of the velocity increment exist (there is considerable experimental evidence for this assumption), the self-similarity hypothesis implies scaling laws for structure functions of arbitrary order:

$$S_p(l) \equiv \langle (\delta v_\parallel(\boldsymbol{r}, l))^p \rangle = C_p \, \epsilon^{\frac{1}{3}p} \, l^{\frac{1}{3}p}. \tag{6}$$

The presence of the factors $\epsilon^{\frac{1}{3}p}$ in the right-hand side ensures that the $C_p$s are dimensionless. The $C_p$s cannot depend on the Reynolds number, since the limit of infinite Reynolds number is assumed. For $p = 3$, it follows from (4) that $C_3 = -\frac{4}{5}$, which is clearly universal. All the $C_p$s, except for $p = 3$, must, however, depend on the detailed geometry of the production of turbulence. In other words, they cannot be universal.

For p=2 the second-order structure function has the same dimension as turbulent kinetic energy

We can interpret (6) as a dependence of kinetic energy on the scale $l$, provided that $l$ is much smaller that $l_0$ (length scale of energy-containing eddies).

Hence, kinetic energy on scale $l$ depends on $l^{2/3}$. .

Podejście alternatywne w języku dekompozycji na szereg Fouriera i liczb falowych (już nie Frisch).

For a flow which is homogeneous in space (i.e. statistical properties are independent of position), a spectral description is very appropriate, allowing us to examine properties as a function of wavelength. The total kinetic energy, given by

$$E = 1/2 \int u_i(\mathbf{x}) u_i(\mathbf{x}) d\mathbf{x} \tag{4.5}$$

can be written in terms of the spectrum $\phi_{i,j}(\mathbf{k})$

$$E = \frac{1}{2} \int \phi_{i,i}(\mathbf{k}) d\mathbf{k} = \int E(\mathbf{k}) d\mathbf{k} \tag{4.6}$$

where $\phi_{i,j}(\mathbf{k})$ is the Fourier transform of the velocity correlation tensor $R_{i,j}(\mathbf{r})$:

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int exp(-i\mathbf{k}.\mathbf{r}) R_{i,j}(\mathbf{r}) d\mathbf{r} \; ; \; R_{i,j}(\mathbf{r}) = \int u_j(\mathbf{x}) u_i(\mathbf{x}+\mathbf{r}) d\mathbf{x} \tag{4.7}$$

$R_{i,j}(\mathbf{r})$ tells us how velocities at points separated by a vector $\mathbf{r}$ are related. If we know these two point velocity correlations, we can deduce $E(\mathbf{k})$. Hence the energy spectrum has the information content of the two-point correlation.

Uwaga: zauważ, że 4.7 zawiera prędkości w punkcie x i x+r, podobnie jak funkcje struktury, zauważ także że pod całką 4.5 jest kwadrat prędkości.

$E(\mathbf{k})$ contains directional information. More usually, we want to know the energy at a particular scale $k = \sqrt{\mathbf{k}.\mathbf{k}}$ without any interest in separating it by direction. To find $E(k)$, we integrate over the spherical shell of radius $k$ (in 3-dimensions):

$$E = \int E(\mathbf{k})d\mathbf{k} = \int_0^\infty \oint E(\mathbf{k})d\sigma dk = \int_0^\infty E(k)dk \tag{4.8}$$

Then

$$E(k) = \oint E(\mathbf{k})d\sigma = \frac{1}{2}\oint \phi_{i,i}(\mathbf{k})d\sigma \tag{4.9}$$

Assuming isotropy:

$$E(k) = 2\pi k^2 \phi_{i,i}(k) \tag{4.10}$$

where $\phi_{i,i}(\mathbf{k}) = \phi_{i,i}(k)$ for all $\mathbf{k}$ such that $\sqrt{\mathbf{k}.\mathbf{k}} = k$.

Równanie na bilans energii w przestrzeni fazowej.

We have an equation for the evolution of the total kinetic energy $E$. Equally interesting is the evolution of $E(k)$, the energy at a particular wavenumber $k$. This will include terms which describe the transfer of energy from one scale to another, via nonlinear interactions.

To obtain such an equation we first take the Fourier transform of the non-rotating, unstratified Boussinesq equations, using the following information about Fourier transforms:

| Physical space | Fourier space |
|---|---|
| $f_i(\mathbf{x}, t)$ | $\hat{f}_i(\mathbf{k}, t)$ |
| $\partial f / \partial x_i$ | $ik_i \hat{f}$ |
| $\nabla f$ | $i\hat{f}\mathbf{k}$ |
| $\nabla^2 f$ | $-k^2 \hat{f}$ |
| $f(\mathbf{x}, t)g(\mathbf{x}, t)$ | $[\hat{f} * \hat{g}]$ |

where $[\hat{f} * \hat{g}] = \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{f}(\mathbf{p}, t)\hat{g}(\mathbf{q}, t) d\mathbf{p}$

Then the momentum equation in physical space

$$\frac{\partial u_i}{\partial t} - \nu \frac{\partial^2 u_i}{\partial x_j^2} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \tag{4.11}$$

becomes, in fourier space:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{u}_i(\mathbf{k}, t) = -ik_m \left(\delta_{i,j} - \frac{k_i k_j}{k^2}\right) \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_j(\mathbf{p}, t)\hat{u}_m(\mathbf{q}, t) d\mathbf{q} \tag{4.12}$$

(The term on the right hand side is the projection of the Fourier transform of $\mathbf{u}.\nabla\mathbf{u}$ onto the plane perpendicular to $\mathbf{k}$. The F.T. of $\nabla P$ is parallel to $\mathbf{k}$, while $\hat{\mathbf{u}}$ etc are all perpendicular to $\mathbf{k}$.)

The term on the right hand side shows that the nonlinear terms involve triad interactions between wave vectors such that $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

Now to obtain the energy equation we multiply eqn 4.12 by $\hat{u}_i(\mathbf{k}', t)$, similarly write an equation for $\hat{u}_i(\mathbf{k}', t)$ and multiply it by $\hat{u}_i(\mathbf{k}, t)$, and add the two equations together, and integrate over $\mathbf{k}'$ to obtain

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \phi_{i,i}(\mathbf{k}, t) = \text{ Triad interaction terms} \tag{4.13}$$

Making use of eqn 4.10 (i.e. assuming isotropy), we then have

$$\frac{\partial}{\partial t} E(k,t) = T(k,t) - 2\nu k^2 E(k,t) \tag{4.14}$$

where $T(k,t)$ comprises the triad interaction terms. If we examine the integral of this equation over all k

$$\frac{\partial}{\partial t} \int_0^\infty E(k)dk = \int_0^\infty T(k,t)dk - 2\nu \int_0^\infty k^2 E(k)dk \tag{4.15}$$

and note that $-2\nu k^2 E(k)$ is the Fourier transform of the dissipation term $-\nu \nabla \mathbf{u}.\nabla \mathbf{u}$, then we see the familiar equation for the total energy budget eqn 4.2 is recovered only if

$$\int_0^\infty T(k,t)dk = 0 \tag{4.16}$$

Hence the nonlinear interactions transfer energy between different wave numbers, but do not change the total energy.

Now, adding a forcing term to the energy equation in k-space we have the following equation for energy at a particular wavenumber $k$:

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) + F(k,t) - 2\upsilon k^2 E(k,t) \tag{4.17}$$

where $F(k,t)$ is the forcing term, and $T(k,t)$ is the **kinetic energy transfer**, due to nonlinear interactions. The **kinetic energy flux** through wave number $k$ is $\Pi(k,t)$, defined as

$$\Pi(k,t) = \int_k^\infty T(k',t)dk' \tag{4.18}$$

or

$$T(k,t) = -\frac{\partial \Pi(k,t)}{\partial k} \tag{4.19}$$

Now for stationary turbulence

$$2\nu k^2 E(k) = T(k) + F(k) \tag{4.20}$$

If $F(k)$, the forcing, is concentrated on a narrow spectral band centered around a wave number $k_i$, then for $k \neq k_i$,

$$2\nu k^2 E(k) = T(k) \tag{4.21}$$

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In the limit of $\nu \to 0$, $T(k) = 0$. If the dissipation rate

$$\epsilon = \int_0^\infty 2\nu k^2 E(k) dk \tag{4.22}$$

then

$$\epsilon = \int_0^\infty F(k) dk \tag{4.23}$$

so that **the rate of dissipation of energy is equal to the rate of injection of energy**. Now in the limit of $\nu \to 0$, but nonzero $F(k)$, $\epsilon$ must remain nonzero, in order to balance the energy injection. (This is achieved by $\int_0^\infty k^2 E(k) dk \to \infty$). Then we find the energy flux in the limit $\nu \to 0$:

$$\begin{aligned} \Pi(k) &= 0, \ : k < k_i \\ \Pi(k) &= \epsilon : k > k_i \end{aligned} \tag{4.24}$$

Hence at vanishing viscosity, the kinetic energy flux is constant and equal to the injection rate, for wavenumbers greater than the injection wavenumber $k_i$. Hence we have the following scenario: Energy is input at a rate $\epsilon$ at a wavenumber $k_i$, is fluxed to higher wavenumbers at a rate $\epsilon$, and eventually dissipated at very high wavenumbers at a rate $\epsilon$, even in the limit of $\nu \to 0$.

Kolmogorov's 1941 theory for the energy spectrum makes use of the result that $\epsilon$, the energy injection rate, and dissipation rate also controls the flux of energy. Energy flux is independent of wavenumber $k$, and equal to $\epsilon$ for $k > k_i$. Kolmogorov's theory assumes the injection wavenumber is much less than the dissipation wavenumber ($k_i << k_d$, or large Re). In the intermediate range of scales $k_i < k < k_d$ neither the forcing nor the viscosity are explicitly important, but instead the energy flux $\epsilon$ and the local wavenumber $k$ are the only controlling parameters. Then we can express the energy density as

$$E(k) = f(\epsilon, k) \tag{4.25}$$

Now using dimensional analysis:

| Quantity | Dimension | |
|---|---|---|
| Wavenumber $k$ | $1/L$ | |
| Energy per unit mass $E$ | $U^2 \sim L^2/T^2$ | we find |
| Energy spectrum $E(k)$ | $EL \sim L^3/T^2$ | |
| Energy flux $\epsilon$ | $E/T \sim L^2/T^3$ | |

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \tag{4.26}$$

$C_K$ is a universal constant known as the Kolmogorov constant. The region of parameter space in $k$ where the energy spectrum follows this $k^{-5/3}$ form is known as the **Inertial range**. In this range, energy **cascades** from the larger scales where it was injected ultimately to the dissipation scale. The theory assumes that the spectra at any particular $k$ depends only on spectrally local quantities - i.e. has no dependence on $k_i$ for example. Hence the possibility for long-range interactions is ignored.

We can also derive the Kolmogorov spectrum in the following manner (after Obukhov): Define an eddy turnover time $\tau(k)$ at wavenumber $k$ as the time taken for a parcel with energy $E(k)$ to move a distance $1/k$. If $\tau(k)$ depends only on $E(k)$ and $k$ then, from dimensional analysis

$$\tau(k) \sim \left[k^3 E(k)\right]^{-1/2} \tag{4.27}$$

The energy flux can be defined as the available energy divided by the characteristic time $\tau$. The available energy at a wavenumber $k$ is of the order of $kE(k)$. Then we have

$$\epsilon \sim \frac{kE(k)}{\tau(k)} \sim k^{5/2} E(k)^{3/2} \tag{4.28}$$

and hence

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \tag{4.29}$$

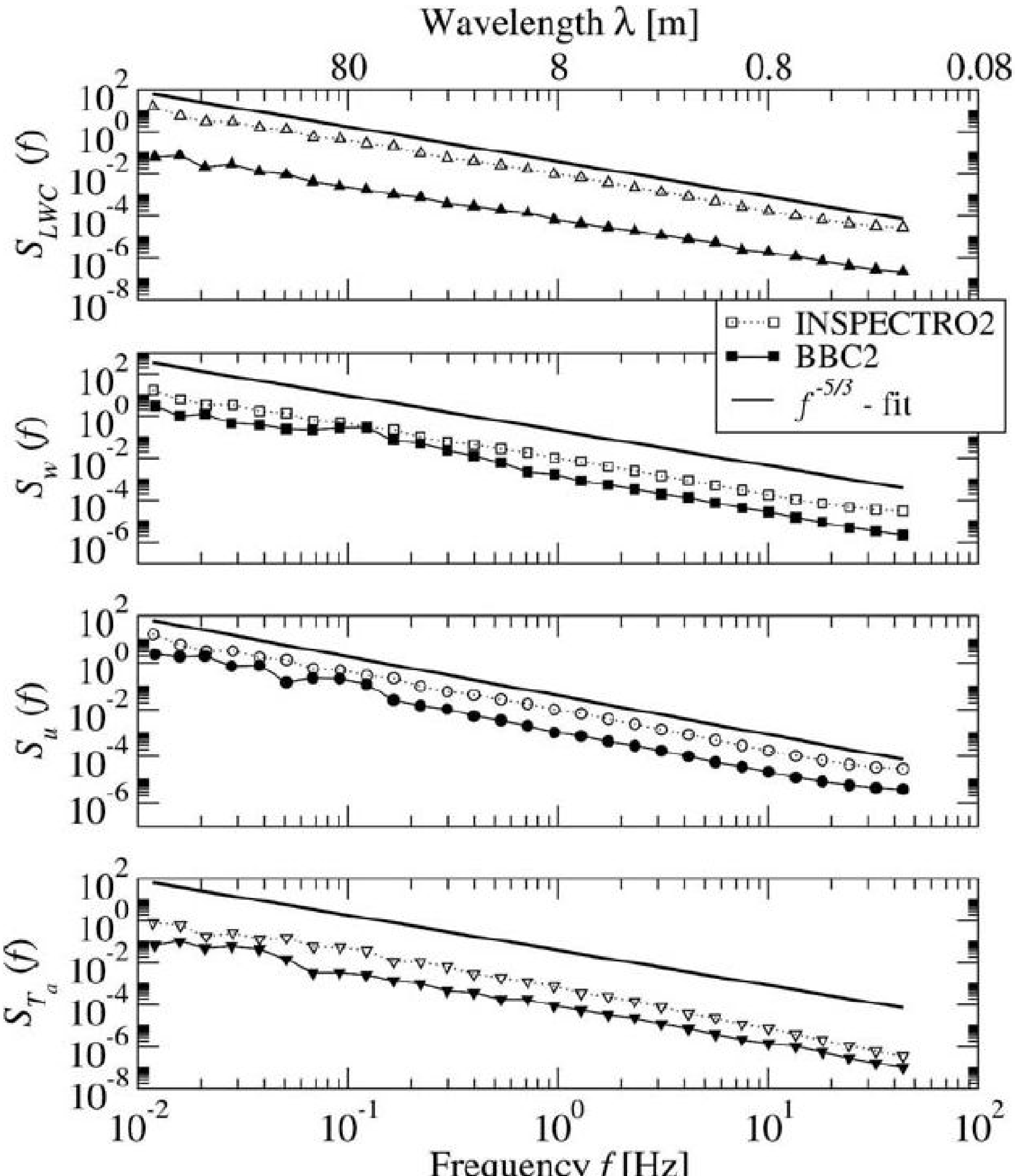
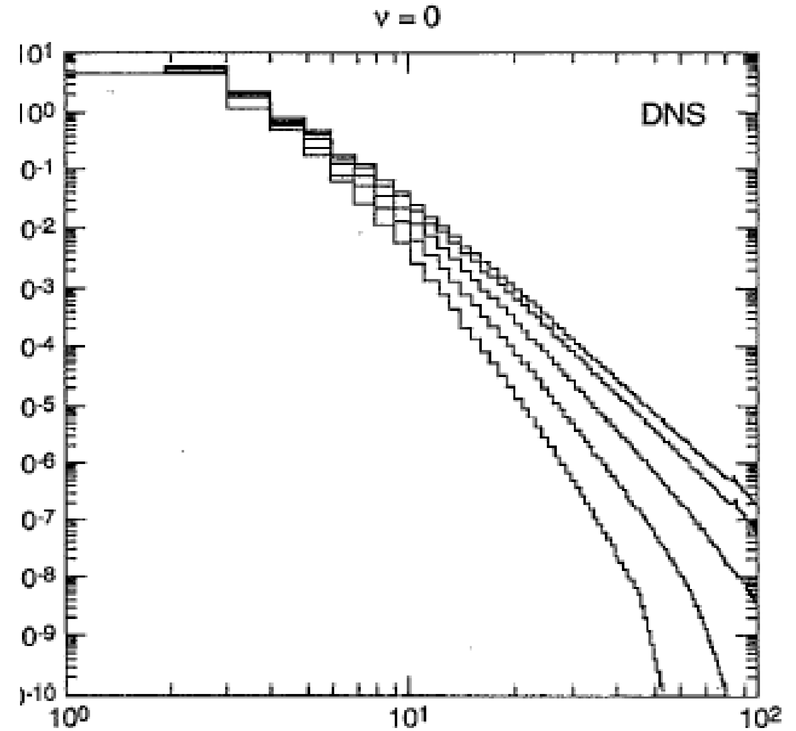



FIG. 5. Power spectral densities $S(f)$ of the same data as presented in Figs. 3 and 4. All spectra are in units of their variance per frequency; spectra of BBC data are divided by a factor of 10 for better resolution. For the top panel the frequencies are converted into wavelength assuming a constant horizontal wind speed of 8 m s$^{-1}$.




. DNS $E(k,t)$ for $t=(0.150, 0.175, 0.200, 0.225, 0.250)$, for initial
on (7), $\nu=0$.

Skala Kołmogorowa i inne charakterystyczne skale turbulencji.

Above a certain wavenumber $k_d$, viscosity will become important, and $E(k)$ will decay more rapidly than in the inertial range. The regime $k > k_d$ is known as the **dissipation range**. An estimate for $k_d$ can be made by assuming

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} : k_i < k < k_d$$
$$E(k) = 0 : k > k_d \tag{4.30}$$

and substituting in eqn 4.22, and integrating between $k_i$ and $k_d$. Then we have

$$k_d \sim \left( \frac{\epsilon^{1/4}}{\nu^{3/4}} \right) \tag{4.31}$$

The inverse $l_d = 1/k_d$ is known as the **Kolmogorov scale**, the scale at which dissipation becomes important.

$$l_d \sim \left( \frac{\nu^{3/4}}{\epsilon^{1/4}} \right) \tag{4.32}$$

Skalę Kołmogorowa często oznaczamy greckim symbolem *η*

At the other end of the spectrum, the important lengthscale is $l_i$, the integral scale, the scale of the energy-containing eddies. $l_i = 1/k_i$. We can also evaluate $l_i$ in terms of $\epsilon$. We can write

$$\overline{u^2} = U^2 = \int_0^\infty E(k)dk \tag{4.33}$$

and substituting for $E(k)$ from eqn 4.26

$$U^2 = \int_0^\infty C_K \epsilon^{2/3} k^{-5/3} dk \tag{4.34}$$

Assume that 1/2 of the energy is contained at scales $k > k_i$. Then

$$U^2 = 6 C_K \epsilon^{2/3} k_i^{-2/3} \tag{4.35}$$

and

$$k_i \sim \frac{\epsilon}{U^3} \tag{4.36}$$

so that $l_i \sim U^3/\epsilon$. Then the ratio of maximum and minimum dynamically active scales

$$\frac{l_i}{l_d} = \frac{k_d}{k_i} \sim \frac{U^3}{\epsilon^{3/4}\nu^{3/4}} \sim \left(\frac{U l_i}{\nu}\right)^{3/4} \sim Re_{l_i}^{3/4} \tag{4.37}$$

where $Re_{l_i}$ is the **Integral Reynolds number**. Hence the range of scales goes as the Reynolds number to the power 3/4. This information is useful in estimating numerical resolution necessary to simulate turbulence down to the Kolmogorov scale at a chosen Reynolds number.

Mikroskala Taylora.

A third length scale often used to characterise turbulence is the **Taylor microscale**:

$$\lambda = \left( \frac{\overline{u_i^2}}{\overline{(\partial u_i / \partial x_j)^2}} \right)^{1/2} = \left( \frac{U^2 \nu}{\epsilon} \right)^{1/2} \qquad (4.38)$$

The Taylor microscale is the characteristic spatial scale of the velocity gradients. Using $\lambda$, an alternative Reynolds number can be defined:

$$Re_\lambda = \frac{U\lambda}{\nu} = \frac{U^2}{\nu^{1/2}\epsilon^{1/2}} \qquad (4.39)$$

where $Re_\lambda \sim Re_{l_i}^{1/2} \sim l_i/\lambda$.

Taylor microscale Reynolds number
liczba Reynoldsa dla mikroskali Taylora.