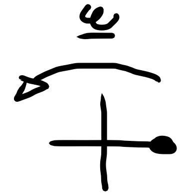
VORTICITY: $\underline{\xi} = \nabla \times \underline{u}$

$$\underline{\xi} = 2\underline{\omega}$$



EULER'S EQ.

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p + \underline{g}$$

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\chi$$

$$\underline{g} = -\nabla\chi \qquad \left(\underline{g} = -g\hat{z}\,,\ \chi = gz\right)$$

1) $\underline{u}\cdot\nabla\underline{u} = \nabla\left(\frac{1}{2}\underline{u}^2\right) + \underbrace{(\nabla\times\underline{u})}_{=\underline{\xi}}\times\underline{u}$

2) $\frac{1}{\rho}\nabla p = \nabla h_0$ $\qquad s = \text{const.} \qquad dh_0 = \frac{1}{\rho}dp$

$$\frac{\partial \underline{u}}{\partial t} + \underline{\xi} \times \underline{u} = -\nabla h_0 - \nabla\left(\frac{1}{2}\underline{u}^2\right) - \nabla \chi$$

$$= -\nabla \underbrace{\left(h_0 + \frac{1}{2}\underline{u}^2 + \chi\right)}_{\equiv \mathcal{H}}$$

$$= -\nabla \mathcal{H}$$

$$\boxed{\frac{\partial \underline{u}}{\partial t} + \underline{\xi} \times \underline{u} = -\nabla \mathcal{H}} \quad (1) \qquad \boxed{\nabla \times \nabla \mathcal{H} = \underline{0}}$$

$$\nabla \times (1) : \quad \frac{\partial}{\partial t} \underbrace{(\nabla \times \underline{u})}_{\underline{\xi}} + \nabla \times (\underline{\xi} \times \underline{u}) = \underline{0}$$

$$\frac{\partial \underline{\xi}}{\partial t} + \underbrace{\nabla \times (\underline{\xi} \times \underline{u})}_{} = \underline{0}$$

$$\nabla \times (\underline{\xi} \times \underline{u}) = \boxed{(\underline{u} \cdot \nabla)\underline{\xi} - (\underline{\xi} \cdot \nabla)\underline{u}} + \cancel{\underline{\xi}(\nabla \cdot \underline{u})} - \cancel{\underline{u}(\nabla \cdot \underline{\xi})}$$

$$\nabla \cdot \underline{\xi} = \nabla \cdot (\nabla \times \underline{u}) = 0$$

Approx. $\nabla \cdot \underline{u} = 0$ (INCOMPRESSIBLE FLOW)

$$\Rightarrow \boxed{\frac{\partial \underline{\xi}}{\partial t} + \underline{u} \cdot \nabla \underline{\xi} = \underline{\xi} \cdot \nabla \underline{u}}$$

$$\frac{\partial \underline{\xi}}{\partial t} + \underline{u} \cdot \nabla \underline{\xi} = \underline{\xi} \cdot \nabla \underline{u}$$

$$\boxed{\frac{d\underline{\xi}}{dt} = \underline{\xi} \cdot \nabla \underline{u}}$$

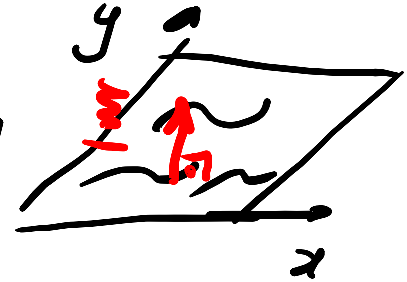
VORTICITY EQ.

FOR $\nabla \cdot \underline{u} = 0$

(HELMHOLTZ EQ.)

IDEAL FLUID

2-D FLOW



$$\underline{u}(x,y) = u(x,y)\,\vec{e}_x + v(x,y)\,\vec{e}_y$$

$$\underline{\xi} = \nabla \times \underline{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{e}_z = \xi\,\vec{e}_z$$

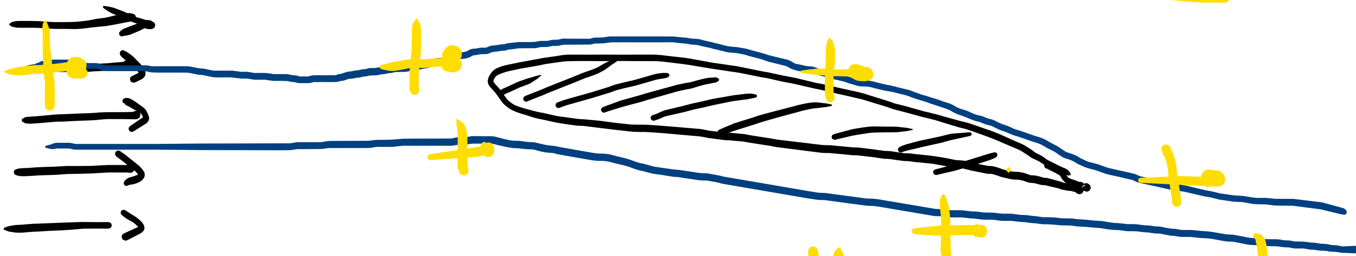
VORTICITY IS INVARIANT

V. EQ. $\quad \dfrac{d\underline{\xi}}{dt} = \underline{\xi} \cdot \nabla u$ $\qquad \boxed{\dfrac{d\underline{\xi}}{dt} = \underline{0}}$

$$(\underline{\xi} \cdot \nabla)\,\underline{u} = (\xi\,\vec{e}_z \cdot \nabla)\,\underline{u} = \xi\,\frac{\partial \underline{u}(x,y)}{\partial z} \equiv \underline{0}$$

WIND TUNNEL CONF.

2D FLOW

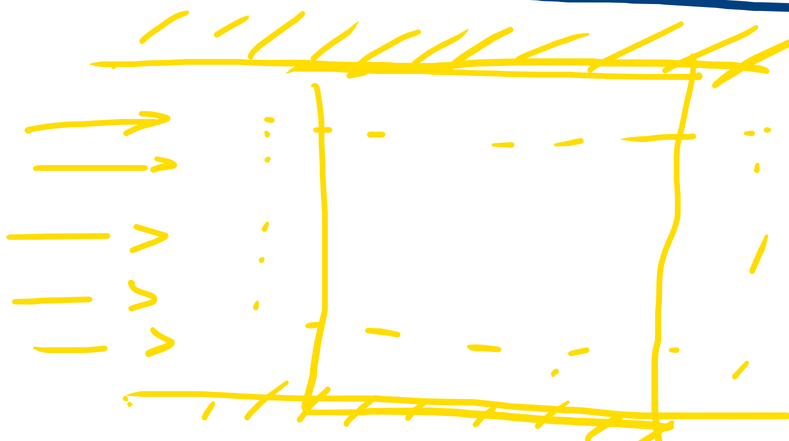


UNIFORM FLOW $\underline{U}$

$\nabla \times \underline{U} = \underline{0}$

FROM $\dfrac{d\underline{\xi}}{dt} = \underline{0}$

$\boxed{\nabla \times \underline{u} = \underline{0} \quad \text{EVERYWHERE}}$



TOP VIEW

$\boxed{\nabla \times \underline{u} = \underline{0}}$ EVERYWHERE

- $\nabla \times \underline{u} = \underline{0} \iff \boxed{\underline{u} = \nabla \phi}$

ϕ : VELOCITY POTENTIAL

- $\nabla \cdot \underline{u} = 0 \Rightarrow \nabla \cdot \nabla \phi = 0$

$$\boxed{\nabla^2 \phi = 0}$$

LAPLACE EQ. FOR THE VEL. POT. ϕ.

$\underline{u}(\underline{r},t)=?$

$p(\underline{r},t)=?$

$\nabla\times\underline{u}=\underline{0}$

DIFFICULT WAY:

$$\frac{\partial\underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}=-\frac{1}{\rho}\nabla p+\underline{g}$$

$$\nabla\cdot\underline{u}=0$$

+

BOUNDARY CONDITIONS.

PROPER WAY

$$\boxed{\frac{d\underline{\xi}}{dt}=\underline{\xi}\cdot\nabla\underline{u}=\underline{0}}$$

2D

$\underline{\xi}(t=0)=\underline{0} \Rightarrow \underline{\xi}(t)=\underline{0}\quad\forall t$

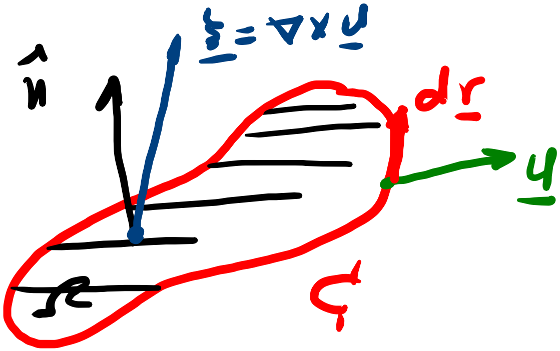
$\nabla\times\underline{u}=\underline{0}\Rightarrow\underline{u}=\nabla\phi$

$\nabla\cdot\underline{u}=0\Rightarrow$

$\nabla^2\phi=0$ + BOUND. COND.

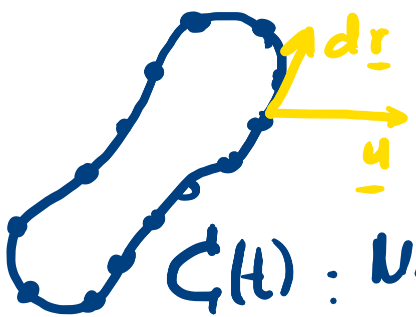
DOES IRROTATIONALITY PERSIST IN 3D FLOWS?

STOKES THEO.



CIRCULATION

$$\oint_C \underline{u} \cdot d\underline{r} = \int_\Omega \underline{\xi} \cdot \hat{n} \, da$$



$C(t)$: MATERIAL CURVE

$$\Gamma(t) = \oint_{C(t)} \underline{u} \cdot d\underline{r}$$

$$\boxed{\frac{d\Gamma(t)}{dt} = ?}$$

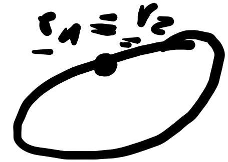
$$\frac{d\Gamma(t)}{dt} = \frac{d}{dt}\oint_{C(t)} \underline{u}\cdot d\underline{r} = \oint_{C} \boxed{\frac{d\underline{u}}{dt}}\cdot d\underline{r} =$$

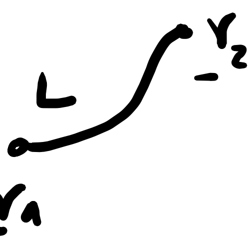
Ex.

$$\frac{du}{dt} = -\frac{1}{\rho}\nabla p - \nabla\chi = -\nabla(h_0+\chi)$$

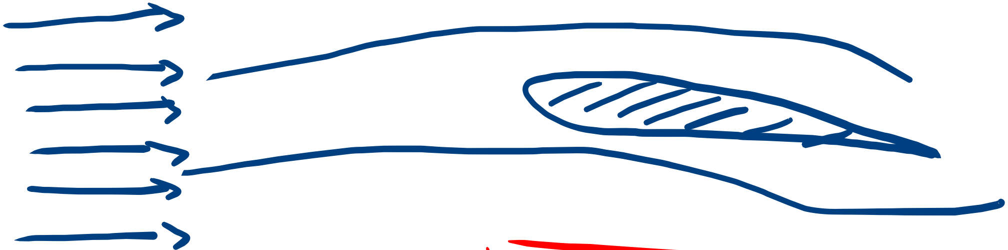
$$\boxed{\frac{d\Gamma(t)}{dt} = 0}$$

$$= \oint_{C} -\nabla(h_0+\chi)\cdot d\underline{r} = -\oint_{C} d(h_0+\chi) = 0$$





$$\int_{\underline{r}_1}^{\underline{r}_2} d(h_0+\chi) = (h_0+\chi)\Big|_{\underline{r}_2} - (h_0+\chi)\Big|_{\underline{r}_1}$$



$\nabla \times \underline{U} = \underline{0}$

$$\boxed{\frac{d\Gamma(t)}{dt} = 0}$$

KELVIN'S CIRCULATION THEO.

FLUID AT REST

$\underline{u} = \underline{0}$

$\nabla \times \underline{u} = \underline{0}$

$\nabla \times \underline{u} = \underline{0}$

BERNOULLI EQUATIONS

$$\frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{u} = -\nabla \mathcal{H}$$

$$\mathcal{H} \equiv h_0 + \frac{1}{2}\underline{u}^2 + \chi$$

BERNOULLI FUNCION

1) STATIONARY FLOW

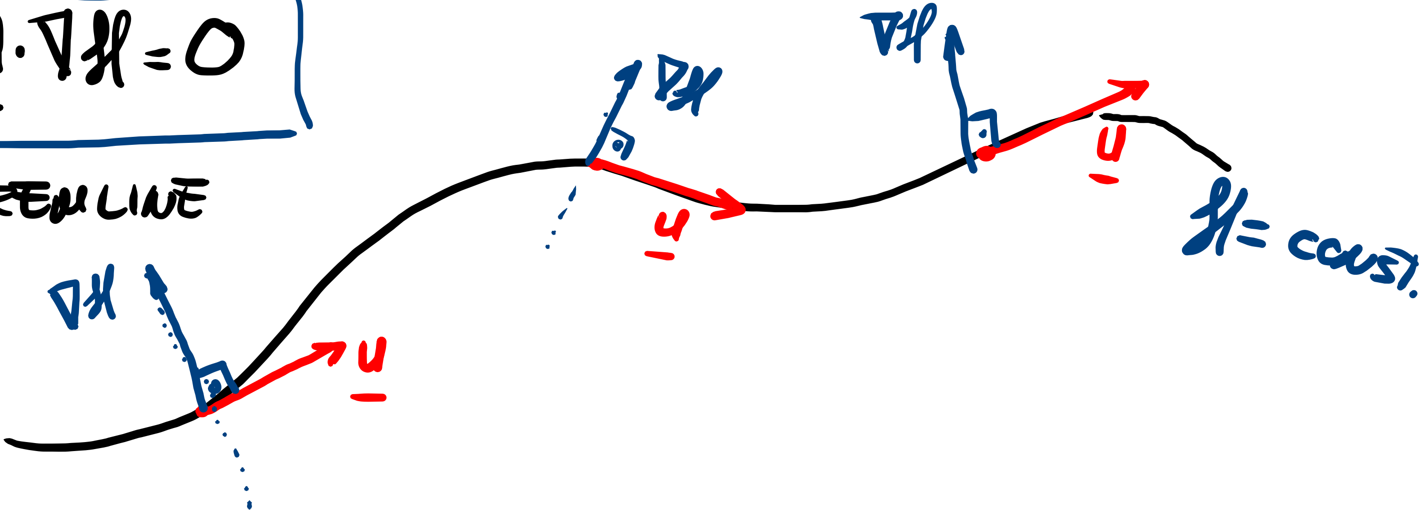
$$\partial_t \underline{u} = \underline{0} \qquad (\nabla \times \underline{u}) \times \underline{u} = -\nabla \mathcal{H} \quad (1)$$

$\underline{u} \cdot (1)$:

$$\boxed{\underline{u} \cdot \nabla \mathcal{H} = 0}$$

$$\boxed{\underline{u}\cdot\nabla\mathcal{H}=0}$$

STREAMLINE



THE BERNOULLI FUNCTION $\mathcal{H}$

IS CONSTANT ALONG A STREAMLINE.

AND

TRAJECTORY

$$\mathcal{H} = h_0 + \frac{1}{2}\underline{u}^2 + \chi = e_0 + \frac{p}{\rho} + \frac{1}{2}\underline{u}^2 + \chi$$

$$\chi = gz$$

IF $\nabla \cdot \underline{u} = 0$, THEN $e_0 =$ CONST.

$$\frac{p}{\rho} + \frac{1}{2}\underline{u}^2 + gz =$$ CONSTANT ALONG STREAMLINES AND TRAJECTORIES.

2)

$$\frac{\partial \underline{u}}{\partial t} + \cancel{\underline{\xi} \times \underline{u}} = -\nabla H$$

IRROTATIONAL FLOW
TRANSIENT

$$\underline{\xi} = \nabla \times \underline{u} = 0$$

$$\partial_t \underline{u} \neq 0$$

$$\nabla \times \underline{u} = \underline{0} \iff \underline{u} = \nabla \phi$$

$$\frac{\partial (\nabla \phi)}{\partial t} + \nabla H = 0$$

$$\nabla \left(\frac{\partial \phi}{\partial t} + H \right) = 0$$

G does not depend on position $\underline{r}$

$$\boxed{\frac{\partial \phi}{\partial t} + H = G(t)}$$

$$\frac{\partial \phi}{\partial t} + \mathcal{H} = G(t) \qquad \underline{u} = \nabla \phi$$

$$\phi(\underline{r}, t) \rightarrow \phi(\underline{r}, t) + \int G(t')\,dt'$$

$$\boxed{\frac{\partial \phi}{\partial t} + \mathcal{H} = 0}$$

3) $\nabla \times \underline{u} = \underline{0}$ AND $\partial_t \underline{u} = 0$

$$\nabla \mathcal{H} = 0$$

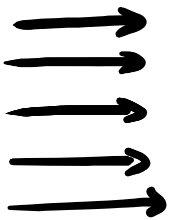
THE BERNOULLI FUNCTION IS UNIFORM

$$\mathcal{H} = h_0 + \frac{1}{2}\underline{u}^2 + \chi = \text{CONST} \quad , \quad h_0 = e_0 + \frac{p}{\rho}$$

$\Rightarrow \nabla \cdot \underline{u} = 0$ THEN $e_0 = \text{CONST.}$ $\quad \chi = gz$

$$\boxed{\frac{p}{\rho} + \frac{1}{2}\underline{u}^2 + gz = \text{CONST.}}$$

$$\boxed{\frac{p}{\rho} + \frac{1}{2}u^2 + \cancel{gz} = \text{CONST.}}$$



$P_\infty, U_\infty,$

$\underline{u}$

$P = \text{CONST} - \rho \frac{1}{2} \underline{u}^2$

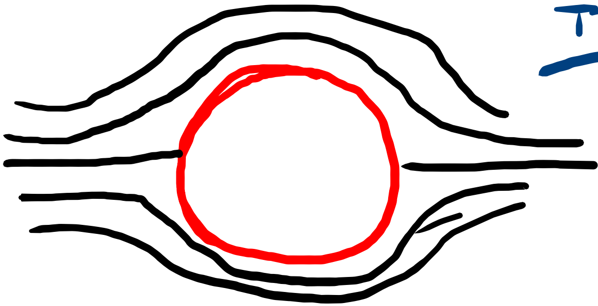
$$\frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 = \text{CONST}$$

NEGLECTING

GRAVITY

2D FLOW

$\underline{u} = ?$, $p = ?$



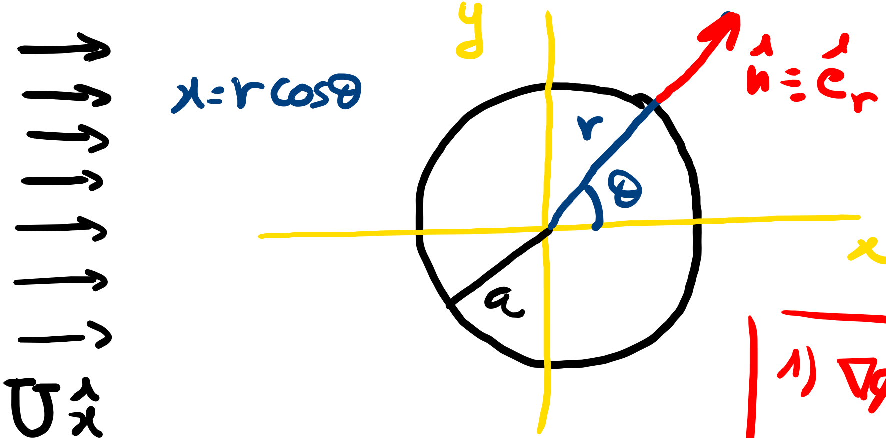
CONFORMAL TRANSF.

FLOW AROUND A CIRCULAR CYLINDER

$\underline{u} = \nabla \phi$

$\nabla^2 \phi = 0$ + B.C.

p FROM BERNOULLI



$\underline{u} = ?$

$$\boxed{\underline{u} = \nabla\phi}$$

$\phi(r,\theta) = ?$

$$\nabla^2\phi = 0$$

1) $\nabla\phi \cdot \hat{n} = \nabla\phi \cdot \hat{e}_r = \boxed{\dfrac{\partial\phi}{\partial r} = 0 \quad r = a}$

2) $\nabla\phi \to U\hat{x}$

$\left(\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}\right) \to (U, 0)$ $\quad \dfrac{\partial\phi}{\partial x} = U$

$\phi = Ux + \cancel{c}$

BOUNDARY CONDITIONS:

1) $\underline{u} \cdot \hat{n} = 0 \quad r = a$ (IMPENETRABLE CYLINDER)

KINEMATIC

2) $\underline{u} \to U\hat{x} \quad r \to \infty$

DYNAMIC

$$\boxed{\begin{matrix}\phi \to Ur\cos\theta \\ r \to \infty\end{matrix}}$$

$\nabla^2\phi = 0$:

$\phi(r,\theta)$

$$\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0}$$

B.C.

$$\frac{\partial\phi}{\partial r} = 0 \qquad r = a$$

$$\phi \to Ur\cos\theta \qquad r \to \infty$$

SEPARATION OF VARIABLES:

$$\phi(r,\theta) = f(r)g(\theta) = \boxed{f(r)\cos\theta = \phi(r,\theta)}$$

ODE FOR $f(r)$:

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} - f = 0$$

$$f(r) = r^n \Rightarrow n = \pm 1$$

$$f(r) = Ar + \frac{B}{r}$$

$A, B = $ CONST.

DEP. ON BOUNDARY COND.

$\phi = f \cos\theta$

$r = a$: $\frac{\partial \phi}{\partial r} = 0 \longrightarrow \frac{df}{dr} = 0$

$r \to \infty$: $\phi \to U r \cos\theta \longrightarrow f \to U r$

$$f = Ar + \frac{B}{r}$$

$$\frac{df}{dr} = 0 \qquad r = a$$

$$f \to Ur \qquad r \to \infty$$

$$\frac{df}{dr} = \left(A - \frac{B}{r^2}\right)_{r=a} = 0$$

$$r \to \infty \quad f \to Ar \qquad \Rightarrow \quad \boxed{\begin{array}{l} A = U \\ B = Ua^2 \end{array}}$$

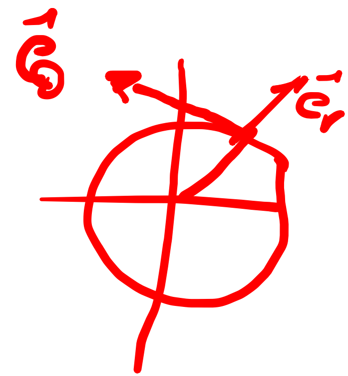
$$\phi = f\cos\theta$$

$$\boxed{\phi(r,\theta) = \left(Ur + \frac{Ua^2}{r}\right)\cos\theta}$$

$$\phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta$$

VELOCITY : $\underline{u} = \nabla\phi = u_r \vec{e}_r + u_\theta \vec{e}_\theta$

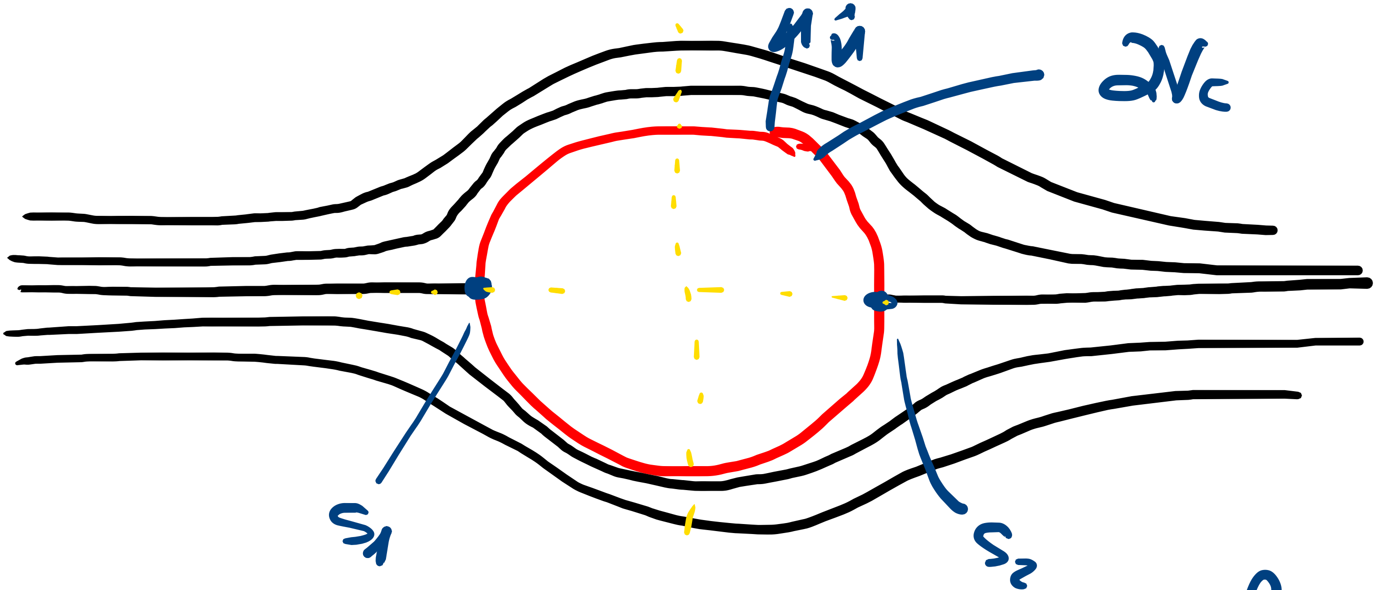
$$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$



$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

FOR $r = a$ (SURFACE)

$$u_r = 0 \quad ; \quad u_\theta = -2U\sin\theta$$



$$\underline{F} = -\int_{\partial V_c} p\,\hat{n}\,dA$$

S_1, S_2 : stagnation points

$$u_r = u_\theta = 0$$

NEXT: PRESSURE: $p = \text{const.} - \frac{1}{2}\rho \underline{u}^2$

$$\underline{F} = -\int_{\partial V_c} p\,\underline{\hat{n}}\,da$$

$$\underline{F} = \underline{0}$$