Exercise Sheet 4

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1. *Hydrostatics of ideal gases*. Derive a relation for the height $z = f(p)$ of a given pressure surface in terms of the pressure p_0 and the temperature T_0 at sea level assuming that the temperature decreases uniformly with height,

   $$T(z) = T_0 - \Gamma z,$$

   where Γ is a positive constant. The relation $z = f(p)$ is the basis for the calibration of aircraft altimeters.

2. Consider an ideal liquid subject to gravity in a cylindrical container rotating in a rigid body motion:

   $$\mathbf{u}(x, y) = (-\Omega y, \Omega x, 0) \tag{1}$$

   Find the shape of the free surface of this liquid, disregarding surface tension. Calculate the difference in height between the center and the border of a cup of coffee of radius $a = 5$ cm, which was stirred at $\Omega = 2\ \mathrm{s}^{-1}$.

3. A fluid in quiescent equilibrium state may be described by:

   $$\mathbf{u} = \mathbf{0}, \quad p = p_{\text{eq}}, \quad \rho = \rho_{\text{eq}}, \quad s = s_{\text{eq}}, \tag{2}$$

   in the absence of gravity for simplicity.

   A sound wave is a small perturbation of the equilibrium state:

   $$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}'(\mathbf{r}, t), \quad p(\mathbf{r}, t) = p_{\text{eq}} + p'(\mathbf{r}, t), \quad \rho(\mathbf{r}, t) = \rho_{\text{eq}} + \rho'(\mathbf{r}, t). \tag{3}$$

   Consider a one-dimensional sound wave and use the expression for $\rho'(x, t)$ in terms of its inverse Fourier transform,

   $$\rho'(x, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho'(k, \omega) \exp[i(kx - \omega t)]\mathrm{d}k\, \mathrm{d}\omega, \tag{4}$$

   as well as the dispersion relation $\omega = \pm ck$ to show that, given

   $$\frac{\partial \rho'}{\partial t}(x, t = 0) = 0 \tag{5}$$

the initial condition will propagate symmetrically in opposite directions:

$$\rho'(x,t) = \tfrac{1}{2} f(x-ct) + \tfrac{1}{2} f(x+ct), \tag{6}$$

where $f(x) = \rho'(x, t=0)$.

4. For an inviscid fluid the Euler's equation reads

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\xi} \times \mathbf{u} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) = -\nabla h_0 - \nabla\Phi, \tag{7}$$

where h_0 is the enthalpy and Φ is the gravitational potential. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\xi}}{\rho}\right) = \left(\frac{\boldsymbol{\xi}}{\rho}\cdot\nabla\right)\mathbf{u}. \tag{8}$$

Hint: use the continuity equation in the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\nabla\cdot\mathbf{u}) = 0. \tag{9}$$