Solutions for final exam

1. Consider waves in the rotating, flat-bottomed shallow water model on an f-plane.

   (a) Linearize the set of equations for this model about the state of rest.

   (b) Assuming plane wave solution of this system obtain the dispersion relation. The phase of the wave can be expressed as $\phi = kx + ly - \omega t$ where $\mathbf{k} = (k, l)$ is wave vector and ω angular frequency.

   (c) Find the group velocity vector of this wave.

   (a) We have the rotating shallow water equations:

   $$\frac{\mathrm{D}\bar{u}}{\mathrm{D}t} + \bar{f} \times \bar{u} = -g\nabla_z h,$$
   $$\frac{\mathrm{D}h}{\mathrm{D}t} + h(\nabla_z \bar{u})) = 0.$$

   Linearization causes us to have an additional term with Coriolis term in the first two equations:

   $$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial \eta'}{\partial x},$$
   $$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial \eta'}{\partial y},$$
   $$\frac{\partial \eta'}{\partial t} + H(\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial x}) = 0.$$

   (b) Using the ansatz for the three variables:

   $$(u, v, \eta) = (\tilde{u}, \tilde{v}, \tilde{\eta})\mathrm{e}^{i(\bar{k}\bar{x} - \omega t)},$$

   we can subsitute it to the equations to get:

   $$\begin{pmatrix} -i\omega & -f_0 & igk \\ f_0 & -i\omega & igl \\ iHk & iHl & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0.$$

   We can calculate the determinant:

   $$\omega(\omega^2 - f_0^2 - c^2K^2) = 0.$$

   This is the dispersion relationship. $c^2 = gH$.

(c) Group velocity is

$$c_x = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}((\pm)\sqrt{f_0^2 + c^2(k^2+l^2)}) = 2c^2k \cdot (\pm)\frac{1}{2}(f_0^2 + c^2(k^2+l^2))^{-1/2}$$

$$c_y = \frac{\partial \omega}{\partial l} = \frac{\partial}{\partial l}((\pm)\sqrt{f_0^2 + c^2(k^2+l^2)}) = 2c^2l \cdot (\pm)\frac{1}{2}(f_0^2 + c^2(k^2+l^2))^{-1/2}$$

Score: 2 points for a), 2 points for b), 1 point for c).

2. Consider a barotropic zonal flow at 32^oN which is east-directed. The flow encounters a north-south-oriented mountain barrier. Before striking the mountain, the westerly wind increases linearly toward the north at a rate of 2 m/s per 500 km. The crest of the mountain range is 2.5 km high and the tropopause, located at 10 km, remains undisturbed.

   (a) What is the initial relative vorticity of the air?

   (b) What is its relative vorticity when it reaches the crest if it is deflected 3^o latitude toward the south during the forced ascent?

   (c) Supposing the current had uniform speed of 13 m/s while crossing crest, what would be the radius of curvature of the streamlines? Assume that while crossing, the wind follows a portion of circular pattern.

   (d) At what latitude would the air need to cross the mountains in order to keep its original relative vorticity?

   (e) Sketch the trajectory of the wind assuming that the initial relative vorticity is equal to zero with comments on the sign of the relative vorticity throughout the crossing process. Think about what is the sign of the relative vorticity for a circular flow clockwise or counter-clockwise.

   (a) Our initial velocity is $\bar{u} = [u(y), 0]$. Therefore:

$$\zeta_i = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = -\frac{2}{500000} = -4 \cdot 10^{-6}\frac{1}{\mathrm{s}}$$

   (b) We consider the conservation of potential vorticity, where everything changes:

$$\zeta_f = \frac{H_f}{H_i}(\zeta_i + f_i) - f_f = \frac{7.5}{10}(-4\cdot 10^{-6} + 2\Omega\sin(32)) - 2\Omega\sin(29) = -1.57\cdot 10^{-5}1/\mathrm{s}$$

   (c)

$$\zeta_f = \omega = \frac{v_f}{R},$$

$$R = \frac{v}{\zeta_f} = \frac{13\mathrm{m/s}}{-1.57\cdot 10^{-5}1/\mathrm{s}} = 8.258\cdot 10^5\mathrm{m} = 825.8\cdot 10^2\mathrm{km}$$

(d)

$$\zeta_i = \zeta_f = \frac{H_f}{H_i}(\zeta_i + f_i) - f_f,$$

$$f_f = \frac{H_f}{H_i}(\zeta_i + f_i) - \zeta_i,$$

$$2\Omega \sin(\phi_f) = \frac{H_f}{H_i}(\zeta_i + 2\Omega \sin(\phi_i)) - \zeta_i,$$

$$\phi = \arcsin(\frac{\frac{H_f}{H_i}(\zeta_i + 2\Omega \sin(\phi_i)) - \zeta_i}{2\Omega} = 23.84^\circ$$

(e) If the air had no relative vorticity in the beginning, then while crossing the barrier, due to decrease in height, the vorticity will decrease. Since it was zero initially, then it will be negative while climbing up the slope. This will curve the wind downwards, towards smaller latitudes. Therefore the Coriolis parameter will be smaller, and the height will be larger, and therefore the relative vorticity will increase to positive values, and the air will surve northward.

Score: 1 point for each subpoint.

3. There is a counterclockwise rotating vortex in cyclostrophic balance, where the tangential velocity is given by the formula:

$$U(R) = U_0 \left(\frac{R}{R_0 + 1}\right)^m \hat{e}_\theta.$$

U_0 and R_0 are initial velocity and initial distance from the center respectively. Calculate the circulation about a streamline at radius R, the vorticity at radius R, and the pressure at radius R. Assume here that P_0 is the pressure at R_0 and that the density is constant. Solution:

$$C = 2\pi RU = 2\pi RU_0 \left(\frac{R}{R_0 + 1}\right)^m = 2\pi U_0 \frac{R^{m+1}}{(R_0 + 1)^m}$$

$$\zeta = -\frac{\partial U}{\partial n} + \frac{U}{R} = \frac{\partial U}{\partial R} + \frac{U}{R} = mU_0 \frac{R^{m-1}}{(R_0 + 1)^m} + U_0 \frac{R^{m-1}}{(R_0 + 1)^m} = (m + 1)\frac{U}{R}$$

Cyclostrophic balance:

$$\frac{U^2}{R} = -\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial R},$$

Integral over R:

$$-\frac{P-P_0}{\rho} = \int_{R_0}^{R} U_0^2 \frac{R^{2n-1}}{(R_0+1)^2 n} \mathrm{d}R = \frac{U_0^2}{(R_0+1)^{2n}} \frac{R^{2n}-R_0^{2n}}{2n},$$

$$P = -\frac{\rho}{2n} \frac{U_0^2}{(R_0+1)^{2n}} (R^{2n}-R_0^{2n}) + P_0$$

1 point for circulation, 2 points for vorticity, 2 points for pressure.

4. Consider the equations of motion in the form

$$\frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} = -\frac{1}{\rho} \nabla p + \bar{F}, \tag{1}$$

$$\nabla \cdot \bar{v} = -\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{2}$$

where **F** denotes friction forces (per unit mass).

(a) Use the vector identities

$$\bar{v} \cdot \nabla \bar{v} = \nabla \left(\frac{1}{2} \bar{v}^2 \right) + \bar{\omega} \times \bar{v},$$

and

$$\nabla \times (\bar{\omega} \times \bar{v}) = (\bar{v} \cdot \nabla) \bar{\omega} - (\bar{\omega} \cdot \nabla) \bar{v} + \bar{\omega} (\nabla \cdot \bar{v}) - \bar{v} (\nabla \cdot \bar{\omega}),$$

to show that:

$$\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} = (\bar{\omega} \cdot \nabla) \bar{v} - \bar{\omega} (\nabla \cdot \bar{v}) + \frac{1}{\rho^2} (\nabla \rho \times \nabla p) + \nabla \times \bar{F}.$$

(b) Simplify the equation above for an inviscid, incompressible and barotropic flow.

(c) Use mass conservation [Eq. (2)] to show that

$$\frac{\mathrm{D}\tilde{\bar{\omega}}}{\mathrm{D}t} = (\tilde{\bar{\omega}} \cdot \nabla) \bar{v} + \frac{1}{\rho^3} (\nabla \rho \times \nabla p) + \frac{1}{\rho} \nabla \times \bar{F},$$

where $\tilde{\bar{\omega}} \equiv \bar{\omega}/\rho$.

(d) Show that, for an adiabatic, inviscid, baroclinic flow,

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0,$$

where

$$Q \equiv \tilde{\bar{\omega}} \cdot \nabla\theta$$

is the *potential vorticity*, and θ is the potential temperature satisfying

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0.$$

Hint: start by multiplying the inviscid form of the last equation from point a) by $\nabla\theta$ from the left side. At the end, consider the orientation of $\frac{1}{\rho^3}\nabla\theta \cdot (\nabla\rho \times \nabla p)$, if $\theta = \theta(p, \rho)$.

(a) Calculate the rotation of the equation (1). Then, use the vector identities and the fact that $\nabla \times (\nabla\phi) = 0$, $\nabla \cdot (\nabla \times \bar{\phi}) = 0$ and $\nabla \times (\alpha\nabla\phi) = \nabla\alpha \times \nabla\phi$. Then consider the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{f(x)} = -\frac{1}{f^2(x)}\frac{\mathrm{d}f}{\mathrm{d}x}$$

(b) Inviscid: $\bar{F} = 0$, incompressible: $\nabla \cdot \mathrm{v} = 0$, barotropic: $\rho = \rho(p)$. The first two assumptions simplify the third and the fifth term. The third assumption is basically saying that the angle between the surface of constant pressure and the surface of constant density is zero.

(c) If you put equation (2) into the third term of the equation from the first exercise, and divide by ρ, then you end up with:

$$\frac{1}{\rho}\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} = \frac{1}{\rho}(\bar{\omega} \cdot \nabla)\bar{v} + \frac{\bar{\omega}}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \frac{1}{\rho^3}(\nabla\rho \times \nabla p) + \frac{1}{\rho}\nabla \times \bar{F},$$

then, comparing to the equation we need to prove, we need to show that:

$$\frac{1}{\rho}\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t} - \frac{\bar{\omega}}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(\frac{\bar{\omega}}{\rho}).$$

It is true, if you apply a rule for integration of the multiplication of two functions.

(d) Multiply the inviscid form of the equation from point a) by $\nabla\theta\cdot$:

$$\nabla\theta\cdot\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t}=\nabla\theta\cdot(\bar{\omega}\cdot\nabla)\bar{v}-\nabla\theta\cdot\bar{\omega}(\nabla\cdot\bar{v})+\nabla\theta\cdot\frac{1}{\rho^2}(\nabla\rho\times\nabla p),$$

rearrange the terms:

$$\nabla\theta\cdot\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t}-\nabla\theta\cdot(\bar{\omega}\cdot\nabla)\bar{v}+\nabla\theta\cdot\bar{\omega}(\nabla\cdot\bar{v})=\nabla\theta\cdot\frac{1}{\rho^2}(\nabla\rho\times\nabla p),$$

the right hand side is equal to zero, because the vector $\nabla\theta$ is perpendicular to the vector $\nabla\rho\times\nabla p$. The LHS is:

$$\nabla\theta\cdot\frac{\mathrm{D}\bar{\omega}}{\mathrm{D}t}-\nabla\theta\cdot(\bar{\omega}\cdot\nabla)\bar{v}+\nabla\theta\cdot\bar{\omega}(\nabla\cdot\bar{v})=0,$$

$$\nabla\theta\cdot(\frac{\partial}{\partial t}\bar{\omega}+(\bar{v}\cdot\nabla)\bar{\omega})-\nabla\theta\cdot(\bar{\omega}\cdot\nabla)\bar{v}+\nabla\theta\cdot\bar{\omega}(\nabla\cdot\bar{v})=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\theta\cdot(\bar{v}\cdot\nabla)\bar{\omega}-\nabla\theta\cdot(\bar{\omega}\cdot\nabla)\bar{v}+\nabla\theta\cdot\bar{\omega}(\nabla\cdot\bar{v})-\nabla\theta\cdot\bar{v}(\nabla\cdot\bar{\omega})=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\theta\cdot(\nabla\times(\bar{\omega}\times\bar{v})\bar{\omega})=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot(\nabla\theta\times(\bar{v}\times\bar{\omega}))-(\bar{v}\times\bar{\omega})\cdot(\nabla\times\nabla\theta)=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot(\nabla\theta\times(\bar{v}\times\bar{\omega}))=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot(\bar{v}(\bar{\omega}\cdot\nabla\theta)-\bar{\omega}(\bar{v}\cdot\nabla\theta))=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot(\bar{v}(\bar{\omega}\cdot\nabla\theta)+\bar{\omega}(\frac{\partial\theta}{\partial t}))=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot\bar{v}(\bar{\omega}\cdot\nabla\theta)+\nabla\cdot(\bar{\omega}\frac{\partial\theta}{\partial t})=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot\bar{v}(\bar{\omega}\cdot\nabla\theta)+\bar{\omega}\nabla\cdot\frac{\partial\theta}{\partial t}=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\nabla\cdot\bar{v}(\bar{\omega}\cdot\nabla\theta)+\bar{\omega}\frac{\partial\nabla\theta}{\partial t}=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\bar{v}\cdot\nabla(\bar{\omega}\cdot\nabla\theta)+(\bar{\omega}\cdot\nabla\theta)\nabla\cdot\bar{v}+\bar{\omega}\frac{\partial\nabla\theta}{\partial t}=0,$$

$$\nabla\theta\cdot\frac{\partial}{\partial t}\bar{\omega}+\bar{v}\cdot\nabla(\bar{\omega}\cdot\nabla\theta)+(\bar{\omega}\cdot\nabla\theta)(-\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t})+\bar{\omega}\frac{\partial\nabla\theta}{\partial t}=0,$$

$$\frac{\mathrm{D}(\bar{\omega}\cdot\nabla\theta)}{\mathrm{D}t}+(\bar{\omega}\cdot\nabla\theta)(-\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t})=0,$$
$$\frac{1}{\rho}\frac{\mathrm{D}(\bar{\omega}\cdot\nabla\theta)}{\mathrm{D}t}-(\bar{\omega}\cdot\nabla\theta)(\frac{1}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t})=0,$$
$$\frac{1}{\rho}\frac{\mathrm{D}(\bar{\omega}\cdot\nabla\theta)}{\mathrm{D}t}-(\bar{\omega}\cdot\nabla\theta)\frac{\mathrm{D}(\frac{1}{\rho})}{\mathrm{D}t}=\frac{\mathrm{D}\frac{(\bar{\omega}\cdot\nabla\theta)}{\rho}}{\mathrm{D}t}=\frac{\mathrm{D}Q}{\mathrm{D}t}=0,$$

Score: 2 points for a), 1 point for b), 1 point for c), 1 point for d).