Scaling laws for turbulence in the atmospheric boundary layer

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Scaling laws for turbulence

Turbulence is characterized by large variations (in time and space) of all variables describing the flow, enhanced mixing, presence of eddies of different sizes and complex interactions between them.

The apparent disorder and chaos can be characterized by means of universal scaling laws for turbulence statistics.

These scaling laws are often related to the invariance of statistics with respect to certain transformations of variables, i.e. the symmetries.

Scaling laws for turbulence

For high Reynolds numbers and far enough from boundaries turbulence is characterized by a tendency to restore the symmetries in the statistical sense.
[U. Frisch *Turbulence: The Legacy of A. N. Kolmogorov*, 2006]

Kolmogorov's $-5/3$ law

$$E(\kappa) \propto \epsilon^{2/3} \kappa^{-5/3}$$

Goal: Investigate the mathematical structure of the underlying transport equations to derive the scaling laws, without solving the equations explicitly.

Diurnal cycle

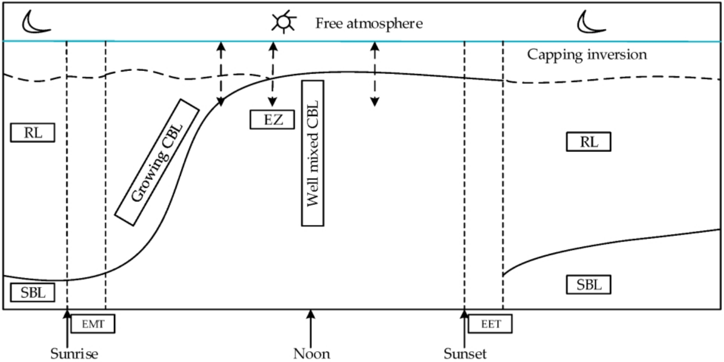

Figure: Diurnal cycle. Image by Dang et al, [Remote Sens. 2019, 11, 1590], licence CC BY 4.0

Stable atmospheric boundary layer

Figure: Arctic. Nasa/Kathryn Hansen, Public domain

Stable atmospheric boundary layer

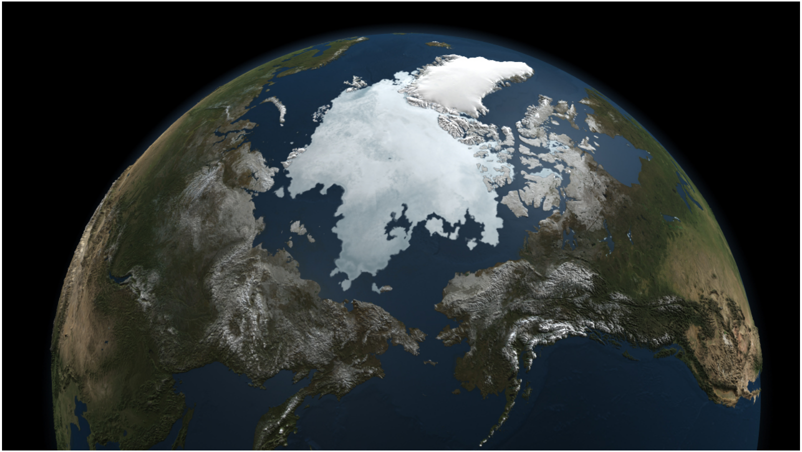

Figure: Arctic. Nasa, Public domain

Monin-Obukhov similarity theory

Basic assumption

Complex vertical structure of the stably-stratified ABL is characterized by a single vertical scale, called the Obukhov scale

$$L = -\frac{1}{\kappa}\frac{|\langle u'w'\rangle_0|^{3/2}}{\langle w'b'\rangle_0}$$

$$b' = g\frac{\theta'}{\bar{\theta}}$$

$\kappa \approx 0.4$ – Kolmogorov constant,
θ' – fluctuating virtual potential temperature,
$\bar{\theta}$ – reference temperature u', w' – fluctuating horizontal and vertical wind velocity components

Monin-Obukhov similarity theory

Non-dimensional momentum and buoyancy gradients

$$\frac{\kappa z}{u_*} S = \phi_m = 1 + 5\,\frac{z}{L},$$
$$\frac{\kappa z}{b_*} N^2 = \phi_h = 1 + 5\,\frac{z}{L},$$

where $S = \mathrm{d}\langle u \rangle / \mathrm{d}z$ is the mean wind shear, and $N^2 = \mathrm{d}\langle b \rangle / \mathrm{d}z$ is the square of the Brunt–Väisälä frequency.

Weak stratifications

$z/L \to 0$ and $\phi_m \to 1$, $\phi_h \to 1$

Strong stratifications

$z/L \gg 1$ and $\phi_m \to 5\frac{z}{L}$, $\phi_h \to 5\frac{z}{L}$

Monin-Obukhov similarity theory

Turbulent Prandtl and Richardson numbers

$$Pr_t = \frac{\phi_h}{\phi_m} = 1, \quad Ri = \frac{N^2}{S^2}$$

Weak stratifications

$z/L \to 0$ and $Pr_t = 1$, $Ri = \frac{z}{L}$

Strong stratifications

$z/L \gg 1$ and $Pr_t = 1$, $Ri = 0.2$

Monin-Obukhov similarity theory - extensions

Local similarity theory by Nieuswtadt (1984)

$$\Lambda = -\frac{1}{\kappa}\frac{|\langle u'w'\rangle|^{3/2}}{\langle w'b'\rangle}$$

$$\langle uw\rangle = \langle uw\rangle_0\left(1-\frac{z}{h}\right)^p, \quad \langle wb\rangle = \langle wb\rangle_0\left(1-\frac{z}{h}\right)^q,$$

Gradient-based scaling by Sorbjan (2006)

$$\langle uw\rangle = \langle w^2\rangle\,\mathcal{G}(Ri), \quad \langle wb\rangle = \langle w^2\rangle N\,\mathcal{H}(Ri).$$

Invariant solutions

Consider the variables z, t, $\theta(z,t)$
Consider new, transformed variables z^*, t^*, $\theta^*(z^*,t^*)$

Invariant

$$C(\theta, z, t) = C(\theta^*, z^*, t^*).$$

Example:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2}$$

$$z^* = \sqrt{\lambda}\, z, \quad t^* = \lambda t, \quad \theta^* = \theta$$

$$\eta = \frac{z^2}{t} = \frac{z^{*2}}{t^*}$$

Invariant solutions

Monin and Obukhov assumed that under the neutral conditions, statistics of relative motions are invariant with respect to the following change of variables

$$z^* = \lambda z, \quad t^* = \lambda t.$$

Monin and Obukhov (1954)

$$\langle u^*(z_2^*)\rangle - \langle u^*(z_1^*)\rangle = \langle u(z_2)\rangle - \langle u(z_1)\rangle = f\left(\frac{z_1}{z_2}\right) \sim \ln\left(\frac{z_1}{z_2}\right)$$

Symmetries of the governing equations

$$\frac{\partial\langle u\rangle}{\partial t}+\frac{\partial\langle uu\rangle}{\partial x}+\frac{\partial\langle uw\rangle}{\partial z}=-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x}$$
$$\frac{\partial\langle w\rangle}{\partial t}+\frac{\partial\langle uw\rangle}{\partial x}+\frac{\partial\langle w^2\rangle}{\partial z}=-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial z}+\langle b\rangle,$$
$$\frac{\partial\langle b\rangle}{\partial t}+\frac{\partial\langle ub\rangle}{\partial x}+\frac{\partial\langle wb\rangle}{\partial z}=0,$$
$$\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle w\rangle}{\partial z}=0.$$

Translation $t^*=t+t_0$, $z^*=z+z_0$ and scaling symmetries of the Navier-Stokes equations

$$t^*=\mathrm{e}^{a_t}t,\quad z^*=\mathrm{e}^{a_z}z,\quad \langle u\rangle^*=\mathrm{e}^{a_z-a_t}\langle u\rangle,\quad \langle b\rangle^*=\mathrm{e}^{a_z-2a_t}\langle b\rangle,$$
$$\langle uw\rangle^*=\mathrm{e}^{2a_z-2a_t}\langle uw\rangle,\quad \langle w^2\rangle^*=\mathrm{e}^{2a_z-2a_t}\langle w^2\rangle,\quad \langle wb\rangle^*=\mathrm{e}^{2a_z-3a_t}\langle wb\rangle$$

Symmetries of the governing equations

Statistical translations Oberlack and Rosteck (2010)

$$\langle u\rangle^* = \langle u\rangle + u_0, \quad \langle b\rangle^* = \langle b\rangle + b_0,$$
$$\langle uw\rangle^* = \langle uw\rangle + uw_0, \quad \langle wb\rangle^* = \langle wb\rangle + wb_0, \quad \langle w^2\rangle^* = \langle w^2\rangle + w_0^2$$

Statistical scaling Oberlack and Rosteck (2010)

$$t^* = t, \quad z^* = z, \quad \langle u\rangle^* = \mathrm{e}^{a_s}\langle u\rangle, \quad \langle b\rangle^* = \mathrm{e}^{a_s}\langle b\rangle,$$
$$\langle uw\rangle^* = \mathrm{e}^{a_s}\langle uw\rangle, \quad \langle w^2\rangle^* = \mathrm{e}^{a_s}\langle w^2\rangle, \quad \langle wb\rangle^* = \mathrm{e}^{a_s}\langle wb\rangle$$

Statistical scaling can represent intermittent laminar-turbulent flow in the SBL

$$\langle u\rangle^* = \mathrm{e}^{a_s}\langle u\rangle + (1 - \mathrm{e}^{a_s})u_0$$

Interpretations

Additional scaling group of temperature at neutral stratifications

$$\langle b\rangle^* = \mathrm{e}^{a_\theta}\langle b\rangle, \quad \langle wb\rangle^* = \mathrm{e}^{a_\theta}\langle wb\rangle$$

Monin-Obukhov invariance of relative motions is recovered for $a_z = a_t$, $a_s = 0$ and $u_0 \neq 0$

$$z^* = \lambda z = \mathrm{e}^{a_z} z, \quad t^* = \lambda t = \mathrm{e}^{a_z} t.$$

$$\langle u^*(z_2^*)\rangle - \langle u^*(z_1^*)\rangle = \langle u(z_2)\rangle - \langle u(z_1)\rangle \sim \ln\left(\frac{z_1}{z_2}\right)$$

Invariant solutions

Solutions of the characteristic system for neutrally buoyant flows with $a_\theta \neq 0$ and $a_s = 0$

$$\frac{\mathrm{d}t}{a_t(t-t_0)} = \frac{\mathrm{d}z}{a_z\,(z-z_0)} = \frac{\mathrm{d}\langle uw\rangle}{(2a_z-2a_t)\langle uw\rangle} =$$
$$= \frac{\mathrm{d}\langle w^2\rangle}{(2a_z-2a_t)\langle w^2\rangle} = \frac{\mathrm{d}\langle u\rangle}{(a_z-a_t)\langle u\rangle + u_0} = \frac{\mathrm{d}\langle \theta\rangle}{a_\theta\langle \theta\rangle + \theta_0}$$

assume that fluxes $\langle uw\rangle = const$, i.e. $a_z = a_t$, $a_\theta = 0$

$$\frac{\mathrm{d}z}{a_z\,(z-z_0)} = \frac{\mathrm{d}\langle u\rangle}{u_0}, \quad \text{hence} \quad \langle u\rangle = \frac{u_0}{a_z}\ln(z-z_0)$$

Oberlack & Rosteck [Discrete Contin. Dyn. Syst. 3, 451 (2010)]

Invariant solutions

Constrain for stratified flows:

$$a_\theta = a_z - 2a_t$$

assumption $\langle uw \rangle = const$, $\langle wb \rangle = const$ leads to $a_z = 0$, $a_t = 0$

$$\frac{\mathrm{d}z}{z_0} = \frac{\mathrm{d}\langle u \rangle}{u_0} = \frac{\mathrm{d}\langle b \rangle}{b_0} = \frac{\mathrm{d}\langle uw \rangle}{0} = \frac{\mathrm{d}\langle w^2 \rangle}{0} = \frac{\mathrm{d}\langle wb \rangle}{0}$$

Linear solution (limit of strong stratifications in the MO theory):

$$\langle u \rangle = \frac{u_0}{z_0} z, \quad \langle b \rangle = \frac{b_0}{z_0} z$$

Yano & Wacławczyk [JAS, 81, (2024)]

Invariant solutions

Solutions of the characteristic system without assumptions about fluxes

$$S = \frac{d\langle u\rangle}{dz} = \tilde{C}_u(X_t)\,(z - z_0)^{\chi-\beta}\,,$$

$$N^2 = \frac{d\langle b\rangle}{dz} = \tilde{C}_b(X_t)\,(z - z_0)^{\chi-2\beta}\,,$$

$$\langle uw\rangle = X_1(X_t)\,(z - z_0)^{2-2\beta+\chi}\,,$$

$$\langle w^2\rangle = X_2(X_t)\,(z - z_0)^{2-2\beta+\chi}\,,$$

$$\langle wb\rangle = X_3(X_t)\,(z - z_0)^{2-3\beta+\chi}\,.$$

where $X_t = (t - t_0)(1 - z/h)^{-\beta}$, $\beta = a_t/a_z$, $\chi = a_s/a_z$.

Summary

- Scaling laws, invariants and symmetries
- Monin Obukhov similarity theory
- Logaritmic solution for in the limit of weak startifications
- Linear solution in the limit of strong stratifications.
- Power-law solution

Measurements

- Surface Heat Budget of the Arctic Ocean (SHEBA) campaign. September 1997– September 1998
- Multidisciplinary drifting Observatory for the Study of Arctic Climate (MOSAiC) expedition. September 2019 – September 2020

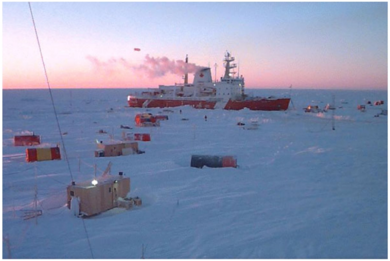

Figure: photo courtesy of the U.S. Department of Energy ARM user facility

Invariant solutions

$$\langle u \rangle = C_u z^{A_u}, \qquad \langle b \rangle = C_b z^{A_b}.$$

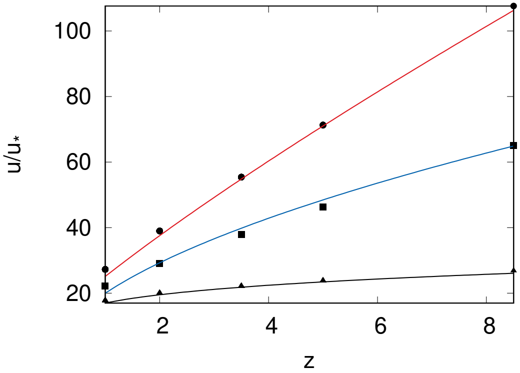

Figure: Scaling of the mean velocity in the surface layer.

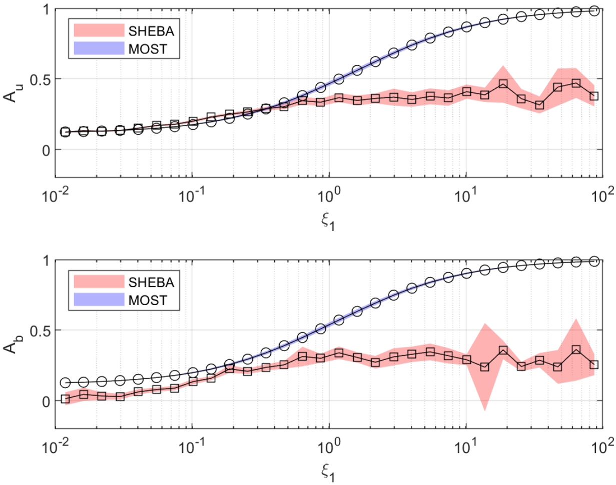

Figure: Exponents a) A_u b) A_b calculated for the theoretical profile (circles) and from SHEBA data (squares) together with 95% confidence intervals. fromWacławczyk et al., BLM 2024

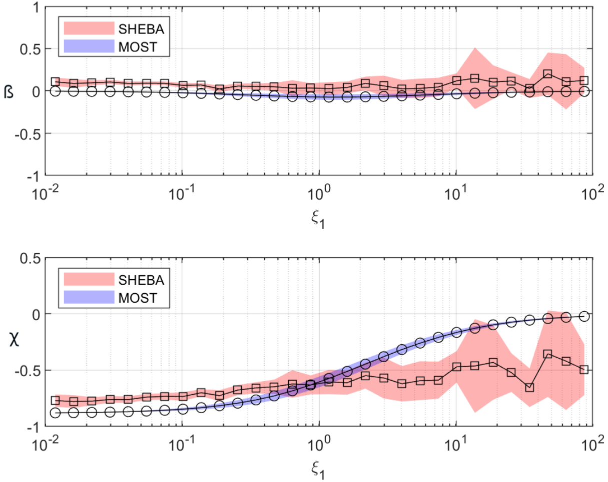

Figure: Same as in previous figure but for exponents β and χ. from Wacławczyk et al., BLM 2024

Local scaling

$$S = \frac{d\langle u\rangle}{dz} = \tilde{C}_u(X_t)\left(1-\frac{z}{h}\right)^{\chi-\beta},$$

$$N^2 = \frac{d\langle b\rangle}{dz} = \tilde{C}_b(X_t)\left(1-\frac{z}{h}\right)^{\chi-2\beta},$$

$$\langle uw\rangle = X_1(X_t)\left(1-\frac{z}{h}\right)^{2-2\beta+\chi},$$

$$\langle wb\rangle = X_3(X_t)\left(1-\frac{z}{h}\right)^{2-3\beta+\chi},$$

$$\frac{\langle uw\rangle}{\langle w^2\rangle} = f(X_t), \tag{1}$$

$$\frac{\langle wb\rangle}{\langle uw\rangle}(t-t_0) = q(X_t). \tag{2}$$

We further assume that (1) and (2) can be inverted.

Invariant solutions

$$\phi_m = \frac{z}{\Lambda}\left(1-\frac{z}{h}\right)^{\chi} F\left(\frac{|\langle uw\rangle|}{\langle w^2\rangle}\right),$$
$$\phi_h = \frac{z}{\Lambda}\left(1-\frac{z}{h}\right)^{\chi} H\left(\frac{|\langle uw\rangle|}{\langle w^2\rangle}\right).$$

where

$$\Lambda = -\frac{1}{\kappa}\frac{|\langle uw\rangle|^{3/2}}{\langle wb\rangle}$$

$$Pr_t = \frac{\phi_h}{\phi_m} = \frac{H}{F} \neq const.$$

For $\chi = 0$ and $F = const$, $H = const$ the linear solution for ϕ_m and ϕ_h is recovered. Moreover $Pr_t = const$ under such conditions.

Wacławczyk et al., BLM 2024

Sorbjan gradient-based theory

The Richardson number

$$Ri = \frac{N^2}{S^2} = \left(1 - \frac{z}{h}\right)^{-\chi} \frac{H\left(\frac{|\langle uw\rangle|}{\langle w^2\rangle}\right)}{F^2\left(\frac{|\langle uw\rangle|}{\langle w^2\rangle}\right)}.$$

For $\chi = 0$ or for small z/h, after inverting the formula for Ri we recover the Sorbjan gradient-based theory

$$\langle uw\rangle = \langle w^2\rangle\, \mathcal{G}(Ri), \quad \langle wb\rangle = \langle w^2\rangle N\, \mathcal{H}(Ri).$$

Data analysis - SHEBA

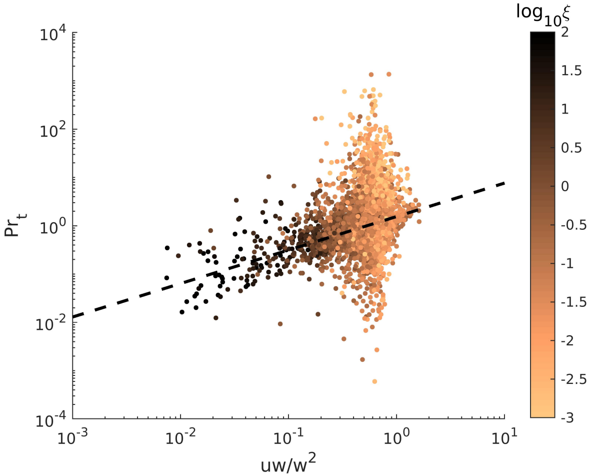

Figure: Pr_t as a function of $\langle uw \rangle / \langle w^2 \rangle$. Color coded is the logarithm $\log_{10}(\xi) = \log_{10}(z_1/L)$. from Wacławczyk et al., BLM 2024

Data analysis - MOSAIC

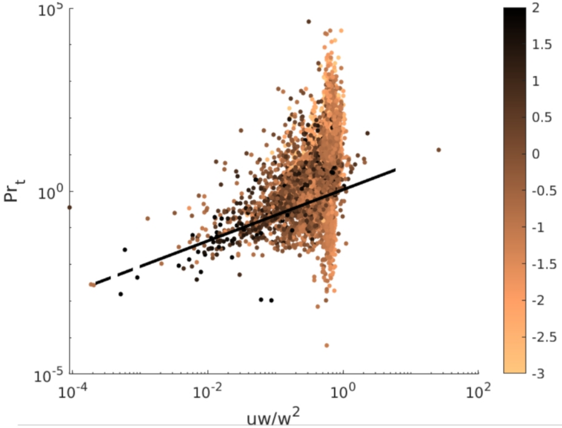

Figure: Pr_t as a function of $\langle uw \rangle / \langle w^2 \rangle$. Color coded is the logarithm $\log_{10}(\xi) = \log_{10}(z_1/L)$.

Similarity functions

$$Pr_t = G\left(\frac{\langle uw\rangle}{\langle w^2\rangle}\right)$$

$$\phi_m = \frac{z}{\Lambda}\frac{1}{Ri}G, \quad \phi_h = \frac{z}{\Lambda}\frac{1}{Ri}G^2,$$

versus

$$\phi_m \propto \left(\frac{z}{\Lambda}\frac{1}{Ri}G\right)^{1/3}, \quad \phi_h \propto \left(\frac{z}{\Lambda}\frac{1}{Ri}G^2\right)^{-1}.$$

which corresponds to logarithmic solutions

Data analysis

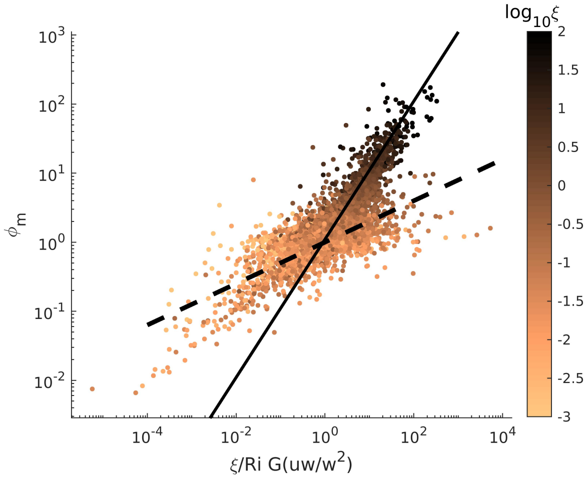

Figure: Similarity function ϕ_m. from Wacławczyk et al., BLM 2024

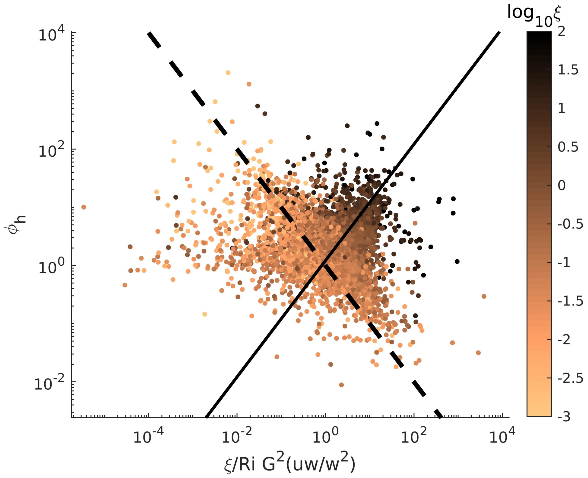

Figure: Similarity function ϕ_h from Wacławczyk et al., BLM 2024

Conclusions

- Linear solution in the stably stratified case was derived based on symmetries of the underlying set of equations.
- The intermittency scaling introduces dependence of the Ri on height.
- The time variability is accounted for through the dependence on the variable $\langle uw \rangle / \langle w^2 \rangle$
- The Pr_t number is not constant but is a function of $\langle uw \rangle / \langle w^2 \rangle$

Perspectives

- Further analysis of the MOSAiC data.
- Including the Coriolis force and horizontal transport into the analysis

Acknowledgements

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