Wave propagation and causality in atmospheric shear layers

Atmospheric physics seminar

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TECHNISCHE UNIVERSITÄT DARMSTADT

DFG SPP 2410

Approximation methods for statistical conservation laws of hyperbolically dominated flow.

MASCHINENBAU
We engineer future

Agenda

1 Problem formulation: Wave propagation in a free shear layer

2 Power series solutions for singular ODEs

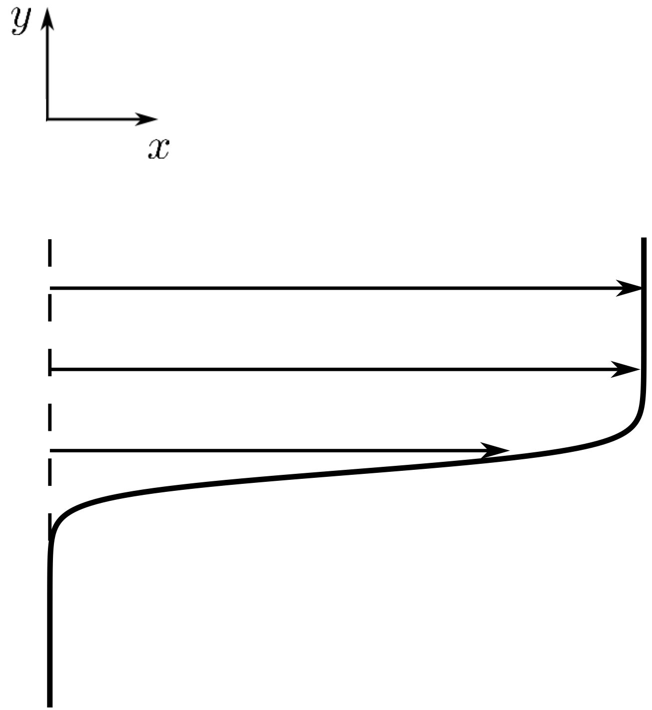
3 Causality to impose uniqueness of solutions

4 Acoustic resonance and wave absorption

5 Extension by means of equivalence transformations

Problem formulation: Wave propagation in a free shear layer

Introduction



$$U_0(y) = \frac{1}{2}(1 + \tanh(y))$$

- Background: Aerospace application & shear flow acoustics
- Focus: Methodology – not the most complete physical model…
- Linear theory to investigate acoustic waves & instabilities

Laplace and Fourier transformation of Euler equations

- decomposition: $p_0 + p(x,y,t), \qquad \rho_0(y) + \rho(x,y,t), \qquad U_0(y)\boldsymbol{e}_y + \boldsymbol{v}(x,y,t)$
- linearized compressible Euler equations: conservation of mass, momentum and energy

$$\frac{D_0\rho}{Dt} + \rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\rho_0 + \rho\boldsymbol{\nabla}\cdot\boldsymbol{V}_0 = 0$$

$$\rho_0\frac{D_0\boldsymbol{v}}{Dt} + \nabla p + \rho_0\boldsymbol{v}\cdot\nabla\boldsymbol{V}_0 + \rho\boldsymbol{V}_0\cdot\boldsymbol{\nabla}\boldsymbol{V}_0 = 0$$

$$\frac{1}{c_0^2}\left(\frac{D_0 p}{Dt} + \boldsymbol{v}\cdot\nabla p_0\right) + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v} + \left(\rho_0\frac{c^2}{c_0^2} + \rho\right)\boldsymbol{\nabla}\cdot\boldsymbol{V}_0 = 0$$

Nondimensionalize: $M = \frac{U_\infty}{c_\infty}$ $\qquad q(x,y,t) = \int_F\int_L \hat{q}(y)e^{i(\alpha x - \omega t)}\mathrm{d}\omega\mathrm{d}\alpha$

$$\boxed{\hat{p}''(y) + \left(\frac{2\alpha U_0'(y)}{\omega - \alpha U_0(y)} + \cancel{\frac{T_0'(y)}{T_0(y)}}\right)\hat{p}'(y) + \left(\frac{M^2\left(\omega - \alpha U_0(y)\right)^2}{\cancel{T_0(y)}} - \alpha^2\right)\hat{p}(y) = 0}$$

isothermal: $T_0 = 1$

Wave reflection and transmission problem

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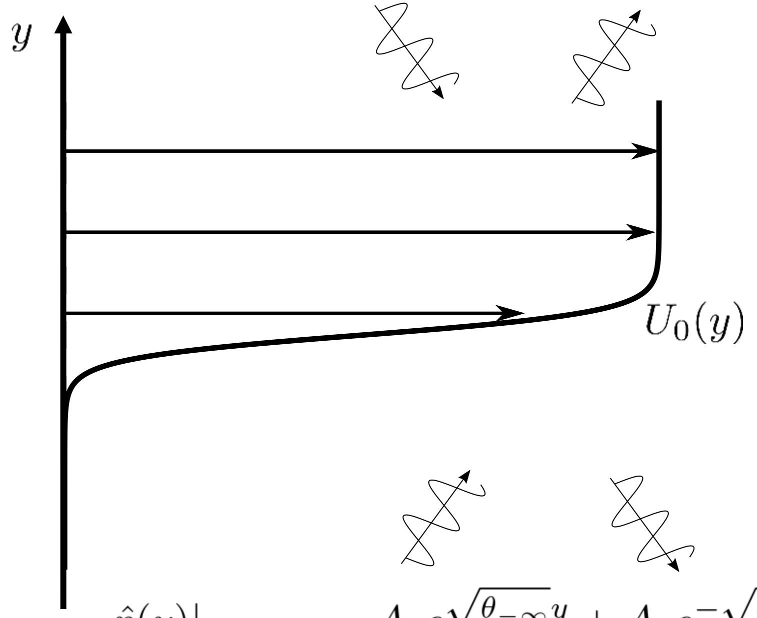
$$\hat{p}(y)|_{y\to\infty} = B_1 e^{-\sqrt{\theta_\infty}y} + B_2 e^{\sqrt{\theta_\infty}y}$$



$$\hat{p}(y)|_{y\to-\infty} = A_1 e^{\sqrt{\theta_{-\infty}}y} + A_2 e^{-\sqrt{\theta_{-\infty}}y}$$

$$\sqrt{\theta_{\pm\infty}} = \sqrt{-M^2(\omega - \alpha U_{\pm\infty})^2 + \alpha^2}, \quad \text{with} \quad U_\infty = 1, \quad U_{-\infty} = 0$$

- neutral modes: streamwise phase velocity $c_x = \dfrac{\omega}{\alpha} \in \mathbb{R}$
- wave-like: solution supersonic w.r.t free stream velocity
  - waves in upper far field: $c_x > 1/M + 1$ or $c_x < 1 - 1/M$
  - waves in lower far field: $c_x > 1/M$ or $c_x < -1/M$
- oscillatory behavior in only one or in both far fields:
  - Wave reflection OR wave reflection and transmission

Far field solutions: Connection problem

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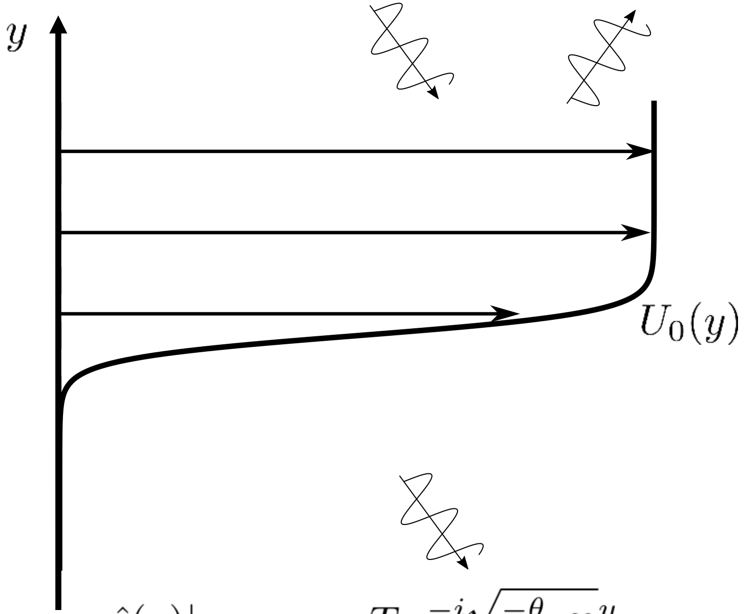
$$\hat{p}(y)|_{y\to\infty} = e^{-i\sqrt{-\theta_\infty}y} + R\,e^{i\sqrt{-\theta_\infty}y}$$



$$\hat{p}(y)|_{y\to-\infty} = T\,e^{-i\sqrt{-\theta_{-\infty}}y}$$

- e.g.: wave incident from upper far field, solution oscillatory in both far fields
- direction of waves by means of Sommerfeld radiation condition
- Unknown: Reflection and transmission coefficients R, T for incident wave with given frequency and wavenumber
- Calculation of reflection and transmission coefficients: Connection of asymptotic far field solutions by full solution of the problem:

$$\hat{p}''(y) + \left(\frac{2\alpha U_0'(y)}{\omega - \alpha U_0(y)}\right)\hat{p}'(y) + \left(M^2\left(\omega - \alpha U_0(y)\right)^2 - \alpha^2\right)\hat{p}(y) = 0$$

Power series solutions for singular ODEs

Frobenius method

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- 2nd order ODEs with singularities: $$\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z} + f(z)\frac{\mathrm{d}w(z)}{\mathrm{d}z} + g(z)w(z) = 0.$$
- Regular singularity at z_s: Analyticity of $(z_s - z)f(z)$ and $(z_s - z)^2\, g(z)$
- Frobenius power series solution: $$w(z, z_s) = C_1 \sum_{n=0}^{\infty} c_{1,n}(z_s - z)^{n+r_1} + C_2 \sum_{n=0}^{\infty} c_{2,n}(z_s - z)^{n+r_2}.$$
  - roots $r_{1/2}$ and recursion formulae for coefficients from inserting into ODE
- Special case: logarithmic solution if $r_2 - r_1 \in \mathbb{R}^+$

$$w(z, z_s) = D_1 \sum_{n=0}^{\infty} d_{1,n}\,(z_s - z)^{n+r_1} + D_2 \left(\sum_{n=0\setminus\{r_1 - r_2\}}^{\infty} d_{2,n}\,(z_s - z)^{n+r_2} + d \sum_{n=0}^{\infty} d_{1,n}\,(z_s - z)^{n+r_1} \ln(z_s - z) \right)$$

- In general: Radius of convergence up to nearest singularity in complex z - plane

Singularities of the PBE

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- Transformation to unit interval by $z = U_0(y) = \frac{1}{2}\left(1 + \tanh(y)\right)$

$$0 = \frac{\mathrm{d}^2\hat{p}(z)}{\mathrm{d}z^2} - \frac{\left((1-2c_x)\,z + c_x\right)}{(c_x - z)\,(z-1)\,z}\frac{\mathrm{d}\hat{p}(z)}{\mathrm{d}z} + \frac{\alpha^2\,(c_x - z)\left((c_x - z)^2 M^2 - 1\right)}{4\,(c_x - z)\,(z-1)^2\,z^2}\hat{p}(z)$$

$\rightarrow \quad z_s = \{0, 1, c_x\}$

- Roots: $z = 0: \quad r_{1/2} = \pm\frac{1}{2}\sqrt{\theta_{-\infty}}$

$z = 1: \quad r_{1/2} = \pm\frac{1}{2}\sqrt{\theta_{\infty}}$

$z = c_x: \quad r_{1/2} = 0,\ 3$

$$w(z, z_s) = C_1 \sum_{n=0}^{\infty} c_{1,n}(z_s - z)^{n+r_1} + C_2 \sum_{n=0}^{\infty} c_{2,n}(z_s - z)^{n+r_2}.$$

- Logarithmic critical layer at $z = c_x = \omega/\alpha \quad \leftrightarrow \quad U_0(y_c) = c_x$
  - Phase velocity of perturbation matches mean flow velocity: Point on physical space for neutral modes
  - Only present as singularity for inviscid considerations → however, also important if viscous effects are considered

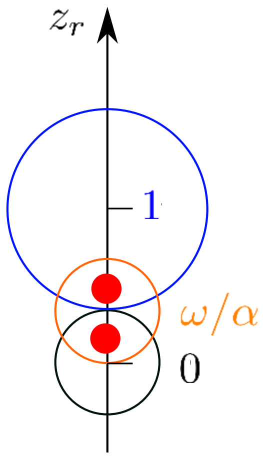
Power series connection for steady solutions

- power series solutions for pressure amplitude $\hat{p}(z)$
- after inserting BCs, solutions matched at ●

- steady nontrivial solution
- amplitude ratios unknown

$$\boxed{\mathbf{M}\cdot\left(\frac{\tilde{A}_2}{\tilde{B}_1},\frac{\tilde{B}_2}{\tilde{B}_1},\frac{D_1}{\tilde{B}_1},\frac{D_2}{\tilde{B}_1}\right)^T=\boldsymbol{b}}$$

$\sim T$ $\sim R$



$$\hat{p}(z,1)=\tilde{B}_1\sum_{n=0}^{\infty}b_{1,n}(1-z)^{n+\frac{1}{2}\sqrt{\theta_\infty}}+\tilde{B}_2\sum_{n=0}^{\infty}b_{2,n}(1-z)^{n-\frac{1}{2}\sqrt{\theta_\infty}}$$

$$\hat{p}(z,c_x)=D_1\sum_{n=0}^{\infty}d_{1,n}\,(z-c_x)^{n+3}+D_2\left(\sum_{n=0\setminus\{3\}}^{\infty}d_{2,n}\,(z-c_x)^{n}+d\sum_{n=0}^{\infty}d_{1,n}\,(z-c_x)^{n+3}\ln\left(z-c_x\right)\right),$$

$$\hat{p}(z,0)=\tilde{A}_1\sum_{n=0}^{\infty}a_{1,n}z^{n-\frac{1}{2}\sqrt{\theta_{-\infty}}}+\tilde{A}_2\sum_{n=0}^{\infty}a_{2,n}z^{n+\frac{1}{2}\sqrt{\theta_{-\infty}}}$$

Causality to impose uniqueness of solutions

Complex conjugation as a (broken?) symmetry

- PBE for neutral waves: ODE with real coefficients → complex-conjugate solutions exist: $\hat{p}(y),\ \hat{p}^*(y)$
- distinguish two cases: waves in **one** (here e.g. upper) far field or in **both** far fields

$$\hat{p}(y)|_{y\to\infty} = B_1 e^{-i\sqrt{-\theta_\infty}y} + B_2 e^{i\sqrt{-\theta_\infty}y}$$
$$\hat{p}(y)|_{y\to-\infty} = A_1 e^{\sqrt{\theta_{-\infty}}y} + \cancel{A_2} e^{-\sqrt{\theta_{-\infty}}y}$$

boundedness

$$\hat{p}(y) \to \hat{p}^*(y)$$

$$\hat{p}^*(y)|_{y\to\infty} = B_1^* e^{i\sqrt{-\theta_\infty}y} + B_2^* e^{-i\sqrt{-\theta_\infty}y}$$
$$\hat{p}^*(y)|_{y\to-\infty} = A_1^* e^{\sqrt{\theta_{-\infty}}y}$$

compatible with BC in lower far field

$$\hat{p}(y)|_{y\to\infty} = B_1 e^{-i\sqrt{-\theta_\infty}y} + B_2 e^{i\sqrt{-\theta_\infty}y}$$
$$\hat{p}(y)|_{y\to-\infty} = \cancel{A_1} e^{i\sqrt{-\theta_{-\infty}}y} + A_2 e^{-i\sqrt{-\theta_{-\infty}}y}$$

only outgoing wave in lower far field

$$\hat{p}^*(y)|_{y\to\infty} = B_1^* e^{i\sqrt{-\theta_\infty}y} + B_2^* e^{-i\sqrt{-\theta_\infty}y}$$
$$\hat{p}^*(y)|_{y\to-\infty} = A_2^* e^{i\sqrt{-\theta_{-\infty}}y}$$

incompatible with BC in lower far field

> complex conjugation broken by BC only for certain streamwise phase velocities

Ambiguity of reflected waves due to complex conjugation

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- Same physical situation described by $\hat{p}(y), \hat{p}^*(y)$

$$\hat{p}(y)|_{y\to\infty} = B_1 e^{-i\sqrt{-\theta_\infty}y} + B_2 e^{i\sqrt{-\theta_\infty}y}$$
$$\hat{p}(y)|_{y\to-\infty} = A_1 e^{\sqrt{\theta_{-\infty}}y}$$

$$R = \frac{B_2}{B_1}$$

$$\hat{p}^*(y)|_{y\to\infty} = B_1^* e^{i\sqrt{-\theta_\infty}y} + B_2^* e^{-i\sqrt{-\theta_\infty}y}$$
$$\hat{p}^*(y)|_{y\to-\infty} = A_1^* e^{\sqrt{\theta_{-\infty}}y}$$

$$R = \frac{B_1^*}{B_2^*} = \left(\frac{B_1}{B_2}\right)^*$$

contradictory?

solutions which are oscillatory in only one field:
Ambiguity of the reflection coefficient due to complex conjugation

→ Which solution branch to choose?

The critical layer and a jump of an invariant

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- Self-adjont PBE by transformation $\tilde{p}(y) = \dfrac{\hat{p}(y)}{c_x - U_0(y)}$

$$\frac{\mathrm{d}^2\tilde{p}(y)}{\mathrm{d}y^2} + F(y)\tilde{p}(y) = 0 \qquad F(y) = \alpha^2\left[(c_x - U_0)^2 M^2 - 1\right] - \frac{2}{(c_x - U_0)^2}\left(\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)^2 - \frac{1}{c_x - U_0}\frac{\mathrm{d}^2U_0}{\mathrm{d}y^2}$$

- $I = \operatorname{Im}\left(\dfrac{\mathrm{d}\tilde{p}}{\mathrm{d}y}\tilde{p}^*\right)$ is an invariant above and below critical layer, shown by considering $\dfrac{\mathrm{d}I}{\mathrm{d}y} = 0$
- Invariant is constant for $y < y_c$ and $y > y_c$ and has a jump at y_c

$$[[I]] = I(y \to \infty) - I(y \to -\infty) = I(y \to y_c^+) - I(y \to y_c^-)$$

- Far field solution can thereby directly be related to jump at critical layer:

$$[[I]] = I(y \to \infty) - I(y \to -\infty) = \frac{\sqrt{M^2(\omega - \alpha)^2 - \alpha^2}}{(c_x - 1)^2}\left(|R|^2 - 1\right) = I(y \to y_c^+) - I(y \to y_c^-)$$

Jump of the invariant from logarithmic solution

- Recall: logarithmic power series solution around critical layer

$$\hat{p}(z, c_x) = D_1 \sum_{n=0}^{\infty} d_{1,n} \left(z - c_x\right)^{n+3} + D_2 \left(\sum_{n=0 \setminus \{3\}}^{\infty} d_{2,n} \left(z - c_x\right)^{n} + d \sum_{n=0}^{\infty} d_{1,n} \left(z - c_x\right)^{n+3} \ln\left(z - c_x\right) \right)$$

- Jump at critical layer: $[[I]] = \mathrm{Im}\left(\frac{\mathrm{d}\tilde{p}}{\mathrm{d}y}\tilde{p}^*\right)\Big|_{z \to z_c^+} - \mathrm{Im}\left(\frac{\mathrm{d}\tilde{p}}{\mathrm{d}y}\tilde{p}^*\right)\Big|_{z \to z_c^-}$

- Using the definition $\tilde{p}(y) = \dfrac{\hat{p}(y)}{c_x - U_0(y)}$ and the logarithmic solution, jump is evaluated

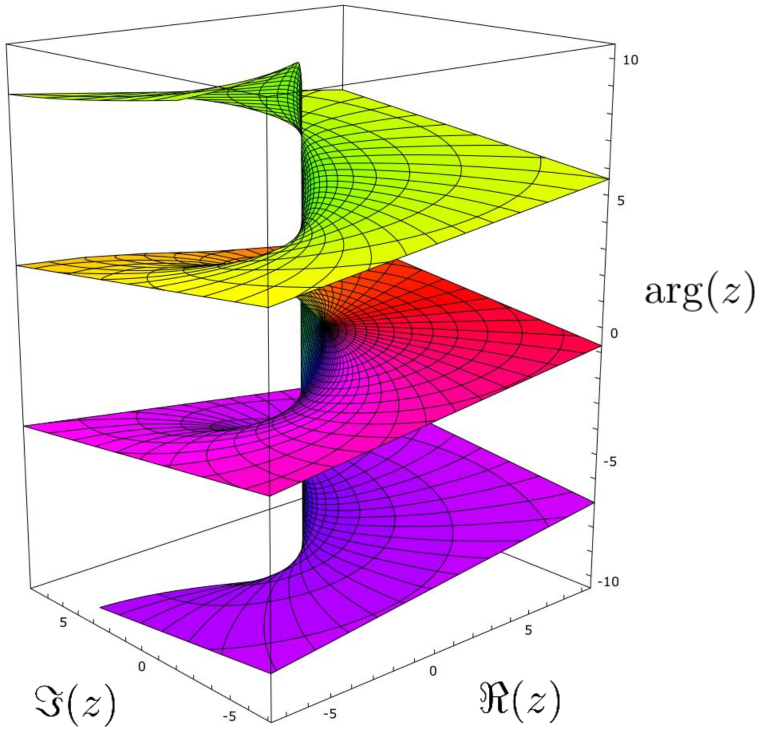
$$[[I]] \approx \frac{(2c_x - 1)\alpha^2}{12(1 - c_x)c_x^3(c_x - 1)^2} \cdot \frac{\mathrm{d}U_0}{\mathrm{d}y}\Big|_{y=y_c} \cdot \left[\mathrm{Im}\left(2\ln(z - c_x) - \ln^*(z - c_x)\right)\right]_{z=c_x^-}^{z=c_x^+}$$

$$[[I]] = \frac{\sqrt{M^2(\omega - \alpha)^2 - \alpha^2}}{\left(c_x - 1\right)^2}\left(\left|R\right|^2 - 1\right)$$

Recall: Branching of complex logarithm

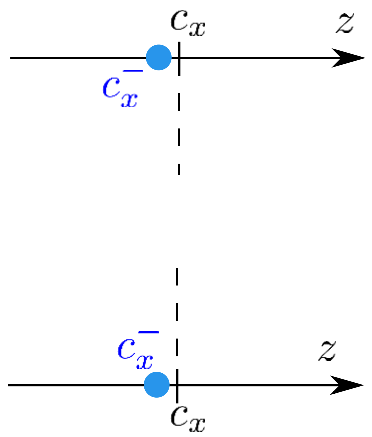
$\mathrm{Im}\,(\ln(z))$? It must hold that $e^{\Re(\ln(z))+i\mathrm{Im}(\ln(z))} = z = |z|\,e^{i\,\arg(z)} \quad \Rightarrow \quad \mathrm{Im}(\ln(z)) = \arg(z)$

→ ambiguous result, since $\arg(z)$ is periodic!



Recall: evaluation of $\mathrm{Im}\,(\ln(z-c_x))\,|_{z\to c_x^-}$

Branch cuts must be introduced for $\ln(z-c_x)$

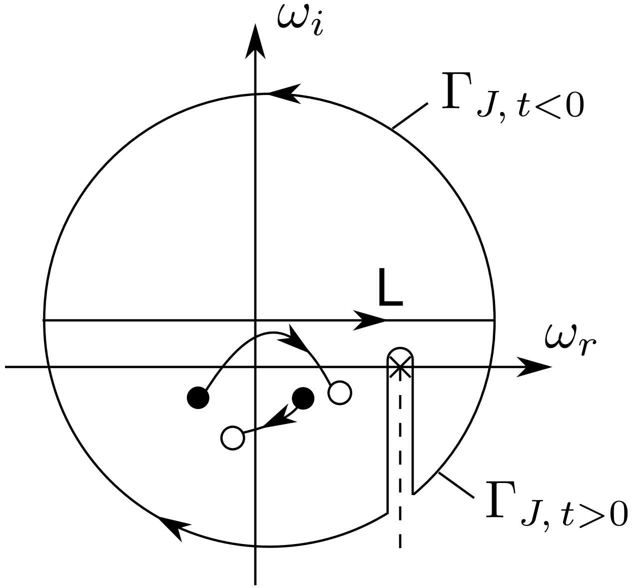


$\arg(0^-) = \pi \;\Rightarrow\; \mathrm{Im}(\ln(0^-)) = \pi$

$\arg(0^-) = -\pi \;\Rightarrow\; \mathrm{Im}(\ln(0^-)) = -\pi$

Causal choice of logarithmic branch cuts

- Recall Laplace transform $p(x,y,t) = \int_F \underbrace{\int_L \hat{p}(\alpha,\omega,y) e^{i(\alpha x - \omega t)}}_{\text{Inverse Laplace contour?}} \mathrm{d}\omega \mathrm{d}\alpha$



- trajectories of unstable eigenvalues along inverse Fourier contour
- Causality requirement: $p(t<0) = 0$
- For $t<0$ we may use the Residual theorem:

$$\int_L \hat{p}(\alpha,\omega,y) e^{-i\omega t} \mathrm{d}\omega + \underbrace{\cancel{\int_{\Gamma_{J,\,t<0}} \hat{p}(\alpha,\omega,y) e^{-i\omega t} \mathrm{d}\omega}}_{\text{Jordan's Lemma}} = 0$$

$\boxed{\text{Causality only fulfilled if log. branch cut is chosen in lower complex half plane}}$ ➡ $\Im(\ln(0^-)) = \pi$

Implications of causality for wave reflection

- With causal choice of logarithmic branch cut:
  - Solutions wave-like in upper free stream $(0 < c_x < 1/2)$

$$\frac{(1-2c_x)\alpha^2 U_0'(y_c)}{4(1-c_x)c_x^3(c_x-1)^2}\pi = \frac{\sqrt{M^2(\omega-\alpha)^2-\alpha^2}}{(c_x-1)^2}\left(|R|^2-1\right)$$

→ Over-reflection of waves in the upper free stream: $|R| > 1$

  - Analogous procedure: Reflected waves in the lower free stream $(1/2 < c_x < 1)$

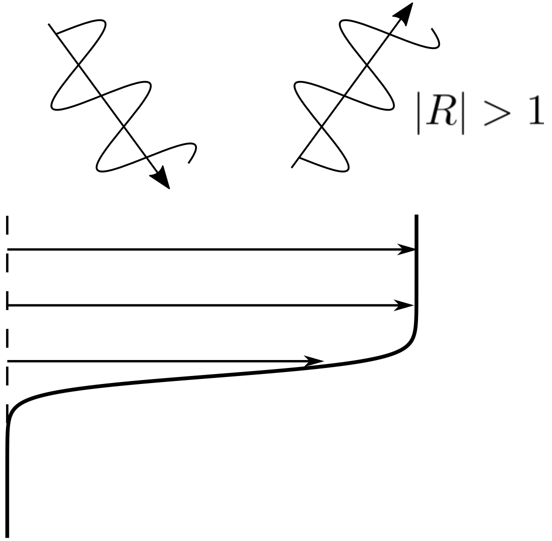
$$\frac{(1-2c_x)\alpha^2 U_0'(y_c)}{4(1-c_x)c_x^3(c_x-1)^2}\pi = \frac{\sqrt{M^2\omega^2-\alpha^2}}{{c_x}^2}\left(|R|^2-1\right)$$

→ Damped reflection of waves in the lower free stream: $|R| < 1$

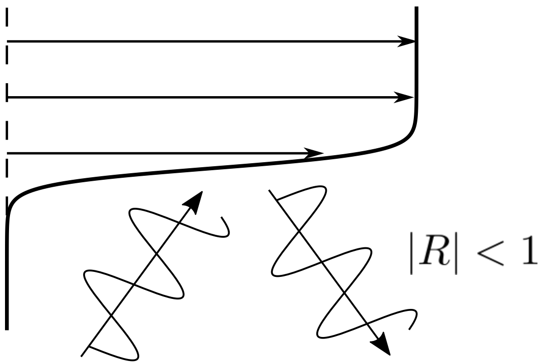
Causality resolves ambiguity of reflection coefficient stemming from complex conjugation

Symmetry?

- Velocity profile is symmetric – how can waves in upper free stream get over relfected but in lower free stream get „under“-reflected?



Incident wave: in direction of decreasing velocity



Incident wave: in direction of increasing velocity

From the perspective of an incident wave: Problem is not symmetric!

Acoustic resonance and wave absorption

Recall: Power series connection

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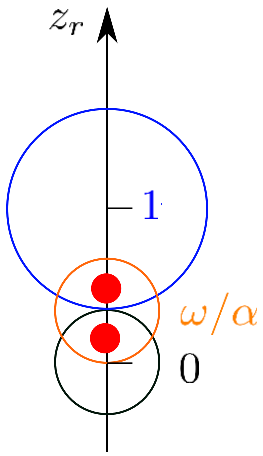
- power series solutions for transformed pressure amplitude $w(z)$
- after inserting BCs, solutions matched at ●
- steady nontrivial solution for all
- amplitude ratios unknown

$$\mathbf{M} \cdot \left(\frac{\tilde{A}_2}{\tilde{B}_1}, \frac{\tilde{B}_2}{\tilde{B}_1}, \frac{D_1}{\tilde{B}_1}, \frac{D_2}{\tilde{B}_1} \right)^T = \boldsymbol{b}$$

$\sim T \qquad \sim R$



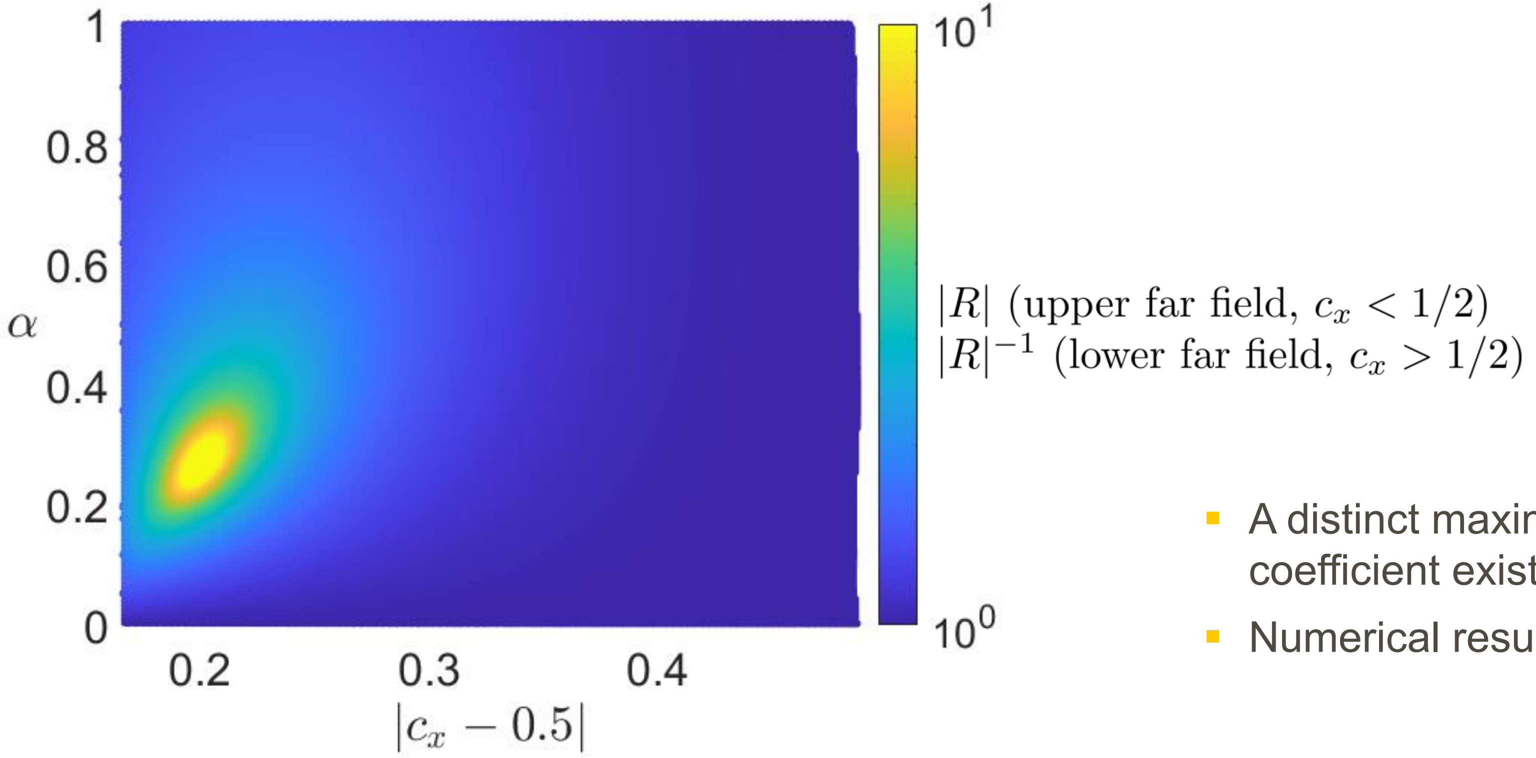
$$\hat{p}(z,1) = \tilde{B}_1 \sum_{n=0}^{\infty} b_{1,n} (1-z)^{n+\frac{1}{2}\sqrt{\theta_\infty}} + \tilde{B}_2 \sum_{n=0}^{\infty} b_{2,n} (1-z)^{n-\frac{1}{2}\sqrt{\theta_\infty}}$$

$$\hat{p}(z,c_x) = D_1 \sum_{n=0}^{\infty} d_{1,n} \left(z-c_x\right)^{n+3} + D_2 \left(\sum_{n=0\setminus\{3\}}^{\infty} d_{2,n} \left(z-c_x\right)^{n} + d \sum_{n=0}^{\infty} d_{1,n} \left(z-c_x\right)^{n+3} \ln\left(z-c_x\right) \right),$$

$$\hat{p}(z,0) = \tilde{A}_1 \sum_{n=0}^{\infty} a_{1,n} z^{n+\frac{1}{2}\sqrt{\theta_{-\infty}}} + \tilde{A}_2 \sum_{n=0}^{\infty} a_{2,n} z^{n-\frac{1}{2}\sqrt{\theta_{-\infty}}}$$

Maximum/ minimum of reflection coefficient

- Numerical results with correct choices of branch cuts for reflected waves



- A distinct maximum/minimum of the reflection coefficient exists
- Numerical results match theoretical prediction

Resonance/Absorption frequency

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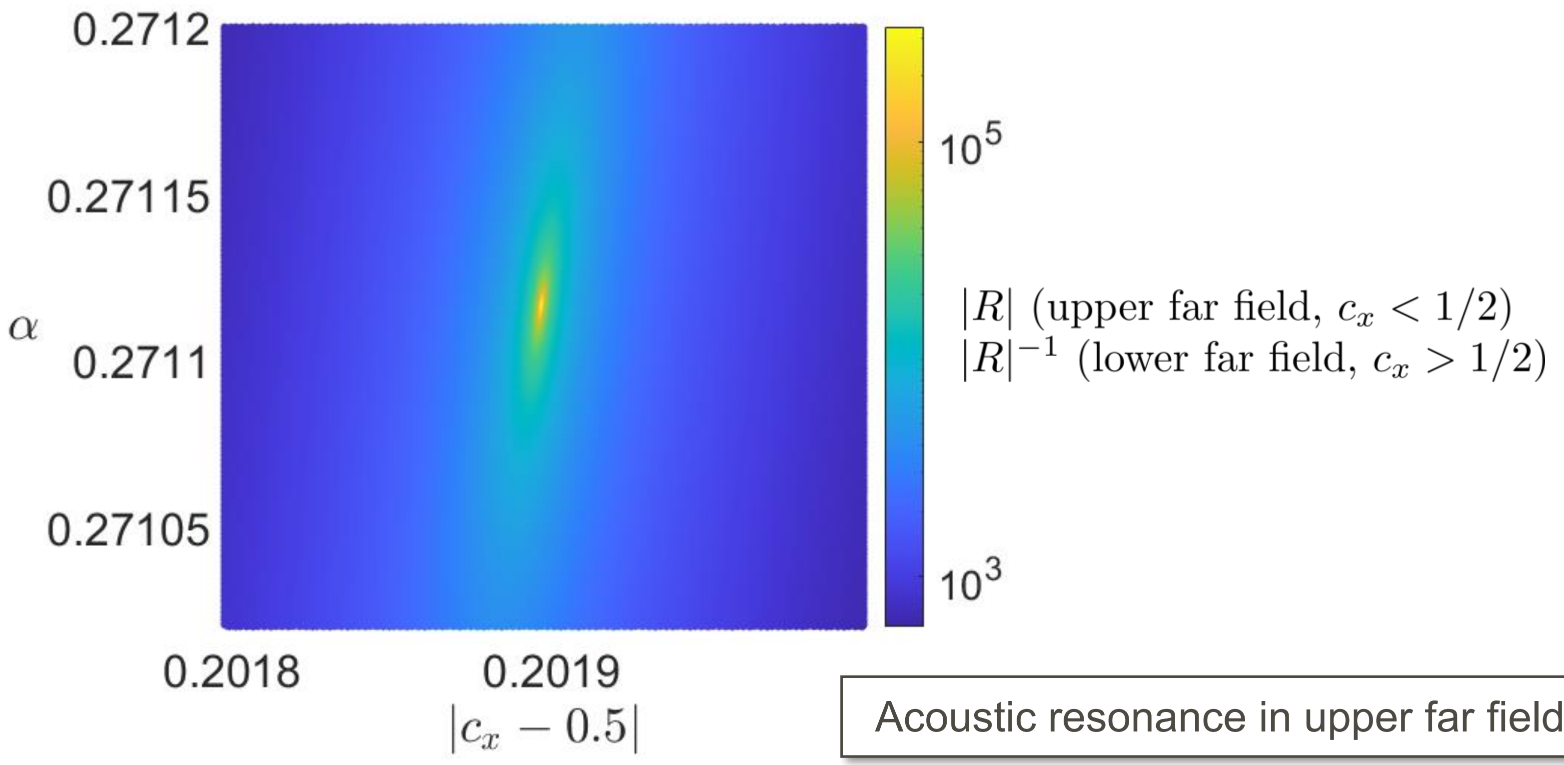
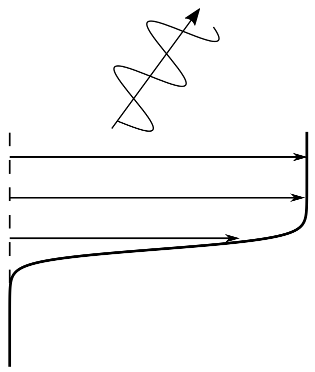
- Finer resolution shows infinite/zero reflection coefficient with a sharp peak



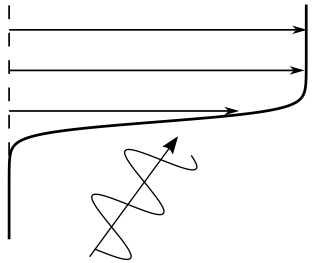
Acoustic resonance in upper far field and wave absorption in lower ff

Resonance and absorption frequencies as eigenvalues

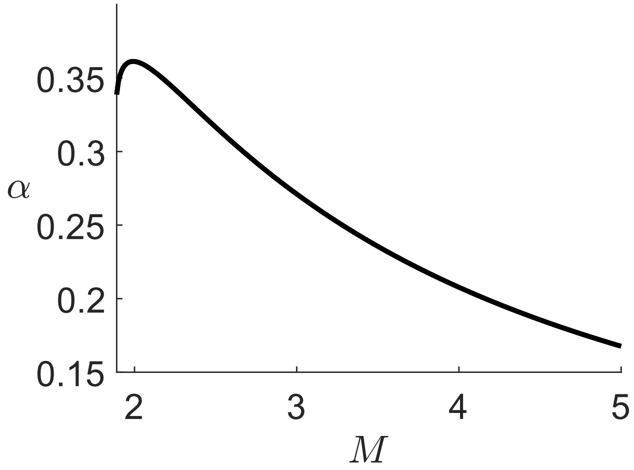
- Problem may also be considered as eigenvalue problem → allow only outgoing or incoming waves

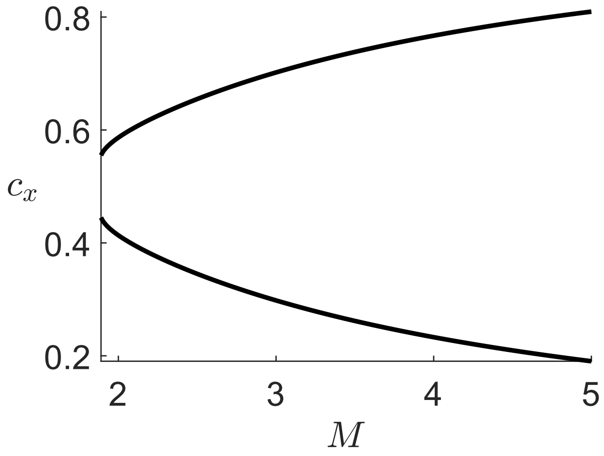


wave radiation



wave absorption





If radiated frequency is excited by incident wave: acoustic resonance

Extension by means of equivalence transformations

Equivalence transformations (I)

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- Aim: Connect two boundary value problems $\mathcal{D}$ (differential operator with parameters & BC)

$$\mathcal{D}(y, \hat{p}(y), \Phi) = 0 \quad \overset{?}{\Leftrightarrow} \quad \tilde{\mathcal{D}}\left(\tilde{y}, \tilde{p}(\tilde{y}), \tilde{\Phi}\right) = 0$$

- One-parameter equivalence transformation $\quad T: \left(\tilde{y}, \tilde{p}, \tilde{\Phi}\right) = T(y, \hat{p}, \Phi, \epsilon)$

$$\tilde{\mathcal{D}}\left(T(y, \hat{p}, \Phi, \epsilon)\right) = 0 \quad \Rightarrow \quad \mathcal{D}(y, \hat{p}(y), \Phi) = 0$$

- Application: 2D/3D solutions:

$$\hat{p}''(y) + \left(\frac{2\alpha_{3\mathrm{D}} U_0'}{\omega_{3\mathrm{D}} - U_0 \alpha_{3\mathrm{D}}}\right) \hat{p}'(y) + \left(M_{3\mathrm{D}}^2 (\omega_{3\mathrm{D}} - U_0 \alpha_{3\mathrm{D}})^2 - (\alpha_{3D} + \beta)^2\right) \hat{p}(y) = 0$$

$$\hat{p}''(y) + \left(\frac{2\alpha_{2\mathrm{D}} U_0'}{\omega_{2\mathrm{D}} - U_0 \alpha_{2\mathrm{D}}}\right) \hat{p}'(y) + \left(M_{2\mathrm{D}}^2 (\omega_{2\mathrm{D}} - U_0 \alpha_{2\mathrm{D}})^2 - \alpha_{2D}^2\right) \hat{p}(y) = 0$$

$$\epsilon = \frac{M_{2\mathrm{D}}}{M_{3\mathrm{D}}}$$

$$\alpha_{3\mathrm{D}} = \pm \epsilon \, \alpha_{2\mathrm{D}}$$

$$\omega_{3\mathrm{D}} = \pm \epsilon \, \omega_{2\mathrm{D}}$$

$$\beta = \pm \sqrt{1 - \epsilon^2} \, \alpha_{2\mathrm{D}}$$

Equivalence transformations (II)

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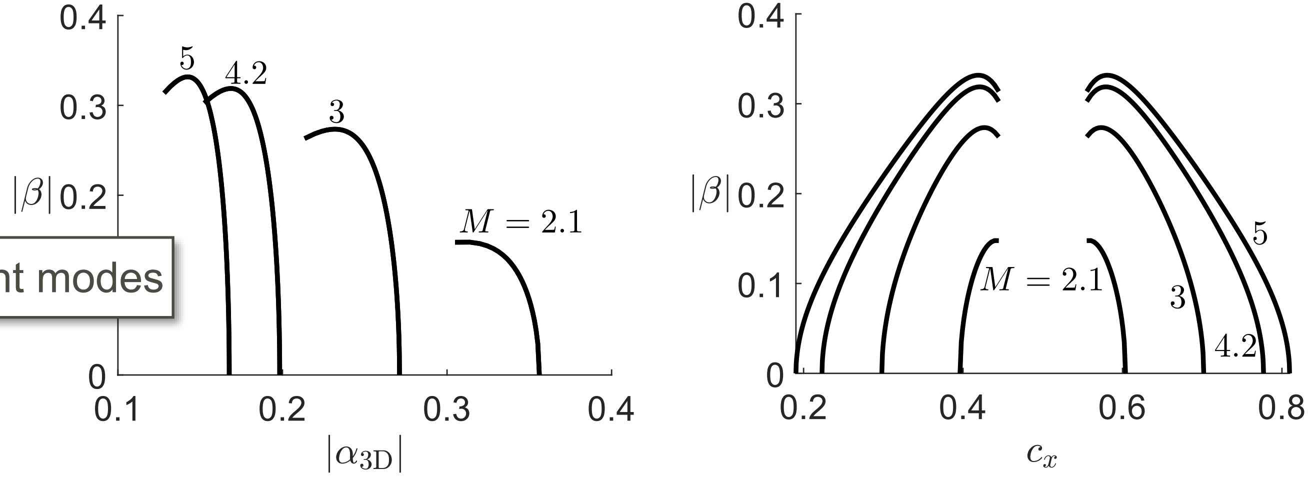
- 3D evolving solutions may be constructed from 2D ones by

$$\alpha_{3\mathrm{D}} = \pm\,\epsilon\,\alpha_{2\mathrm{D}}$$
$$\omega_{3\mathrm{D}} = \pm\,\epsilon\,\omega_{2\mathrm{D}} \qquad \epsilon = \frac{M_{2\mathrm{D}}}{M_{3\mathrm{D}}}$$
$$\beta = \pm\sqrt{1-\epsilon^2}\,\alpha_{2\mathrm{D}}$$

- Consequence: $\epsilon \in [0,1]$ → solution space of 3D solutions at given Mach number is built by all 2D solutions at lower Mach numbers



Multiple obliquely propagating resonant modes

Geophysics seminar?

TECHNISCHE UNIVERSITÄT DARMSTADT

- Linearized Euler including Boussinesq approximation [Cushman-Roisin & Beckers (2006), Introduction to geophysics]

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{d\bar{u}}{dy} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$
$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} - \frac{\rho' g}{\rho_0}$$
$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$
$$\frac{\partial \rho'}{\partial t} + \bar{u}\frac{\partial \rho'}{\partial x} + v'\frac{d\bar{\rho}}{dy} = 0$$

$$N^2 = -\frac{-g/\rho_0}{\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}y}}$$
$$Ri = \frac{N^2}{\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2}}$$

nondim.

$$\bar{u}(y) = \frac{1}{2}(1+\tanh(y))$$
$$\bar{\rho}(y) = -y$$
$$q'(x,y,t) = \int_F\int_L \hat{q}(y)e^{i(\alpha x-\omega t)}\mathrm{d}\omega\mathrm{d}\alpha$$

profiles + NMA

$z = \bar{u}(y)$

Same singularities $0, 1, c_x$: Probably an equivalence transformation is possible – at least: same numerics can be applied

$$0 = \frac{\mathrm{d}^2\hat{\rho}(z)}{\mathrm{d}z^2} - \frac{z^2-((3+c_x)\,z+c_x}{(c_x-z)\,(z-1)\,z}\frac{\mathrm{d}\hat{\rho}(z)}{\mathrm{d}z} + \frac{\alpha^2\,(z-c_x)^2+Ri}{4\,(c_x-z)^2\,(z-1)^2\,z^2}\hat{\rho}(z)$$

Thank you for listening ☺

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