Exercise Sheet 3

Transport of material integrals and balance equations

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1. Show that inclusion of a force density **f** (force per unit volume) in the equation of motion[1],

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \nabla \cdot \boldsymbol{\Sigma} + \mathbf{f}, \tag{1}$$

does not change the main result of angular momentum balance, that is, the stress tensor $\boldsymbol{\Sigma}$ must be symmetric.

2. Show that, in the presence of the gravitational field with acceleration $\mathbf{g} = -\nabla\Phi$, the balance equation for the total energy takes the form:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(e_0 + \tfrac{1}{2}\mathbf{u}^2 + \Phi\right) = \nabla \cdot (\boldsymbol{\Sigma} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}, \tag{2}$$

where e_0 is the internal energy per unit mass, and $\mathbf{q}$ is the heat flux per unit area and per unit time.

3. Prove the basic properties of material fluid elements and transport equations:

Show that the deformation (stretch) of an element $\mathrm{d}\boldsymbol{\ell}$ of a material line in a velocity field $\mathbf{u}(\mathbf{r}, t)$ is

$$\frac{\mathrm{d}(\mathrm{d}\boldsymbol{\ell})}{\mathrm{d}t} = (\mathrm{d}\boldsymbol{\ell} \cdot \nabla)\mathbf{u}. \tag{3}$$

Use this result to show that an integral along a material curve with endpoints $P(t)$ and $Q(t)$ of a scalar function $\theta(\mathbf{r}, t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{P(t)}^{Q(t)} \theta \,\mathrm{d}\boldsymbol{\ell} = \int_P^Q \frac{\mathrm{d}\theta}{\mathrm{d}t}\mathrm{d}\boldsymbol{\ell} + \int_P^Q \theta \,\mathrm{d}\boldsymbol{\ell} \cdot \nabla\mathbf{u}. \tag{4}$$

[1] For example, to account for the presence of the gravitational field, we set $\mathbf{f} = \rho\mathbf{g}$, where $\mathbf{g}$ is the acceleration of gravity.

Then show that an integral over a closed material curve $\mathcal{C}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}(t)} \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} = \oint_{\mathcal{C}} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{5}$$

4. Derive the balance equation for the *linear momentum density* (i.e., linear momentum per unit volume),

$$\mathbf{j}(\mathbf{r}, t) \equiv \rho(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t).$$

Then write the balance equation in the form of a *general continuity equation*:

$$\frac{\partial z}{\partial t} = -\nabla \cdot \mathbf{J}_z + \theta(z), \tag{6}$$

where z is the density of a given quantity, $\mathbf{J}_z$ is the flux of z per unit surface and unit time, and $\theta(z)$ denotes the local production of z per unit time and volume. Provide a physical interpretation of the terms appearing in the continuity equation.