Solutions for midterm exam

1. Given the following expression for the geopotential field

$$\phi = \phi_0(p) + cf\left\{-y\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right] + \frac{\sin(k(x-ct))}{k}\right\}$$

where ϕ_0 is a function of p alone, c is a constant speed, k is a zonal wavenumber and $p_0 = 1000$ hPa, obtain the expression for the corresponding geostrophic wind. What is the gradient of temperature in horizontal directions?

Use the geostrophic balance to obtain the geostrophic wind:

$$\bar{f} \times \bar{u}_g = -\nabla_p\phi,$$

$$\bar{u}_g = \frac{1}{f}(\hat{k} \times \nabla_p\phi),$$

then calculate the $\nabla_p\phi$:

$$\nabla_p\phi = (cf\cos k(x-ct), -cf\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right]),$$

and obtain the geostrophic wind:

$$\bar{u}_g = (c\left[\cos\left(\frac{\pi p}{p_0}\right) + 1\right], c\cos k(x-ct)),$$

then using the thermal wind equation:

$$f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}(\frac{\partial T}{\partial x})_p,$$

$$f_0 \frac{\partial u_g}{\partial p} = \frac{R}{p}(\frac{\partial T}{\partial y})_p,$$

obtain the gradients:

$$(\frac{\partial T}{\partial x})_p = -\frac{p f_0}{R}\frac{\partial v_g}{\partial p} = 0,$$

$$(\frac{\partial T}{\partial y})_p = \frac{p f_0}{R}\frac{\partial u_g}{\partial p} = -\frac{f_0 p c \pi}{R p_0} \sin\left(\frac{\pi p}{p_0}\right).$$

Score: 1 point for the derivative of the geopotential, 2 points for the geostrophic wind, 2 points for the temperature gradient.

2. Consider an ocean region with no interior geostrophic flow subject to a constant wind stress τ_w directed along the x axis. Continuity of momentum flux at the surface and motion assumed to vanish deep ($z \to -\infty$) in the water implies the following profile of the flow velocity:

$$\begin{aligned} u &= \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\gamma z}(\cos\gamma z + \sin\gamma z) \\ v &= \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\gamma z}(\sin\gamma z - \cos\gamma z) \end{aligned}$$

where $\gamma = \sqrt{\frac{f}{2K}}$ and K is eddy viscosity in the ocean.

Assume the beta-plane approximation for the latitude of 43^o N and the typical values of eddy viscosity $K = 10^{-3}$ m^2/s, wind stress over ocean $\tau_w = 0.1$ Pa and constant water density $\rho = 1000$ kg/m^3.

   (a) Find the surface water velocity and its direction with respect to the wind stress. Draw a sketch showing respective vectors.

   (b) Find the highest level where the direction of the flow is opposite to that at the surface.

   (c) Calculate the net mass transport over entire depth (to get proper units consider mass of water crossing in unit time the plane of unit width and infinite height). Find its direction with respect to the wind stress and mark it in the sketch. Explain physically the angle between them.

(d) Estimate the value of $\beta = \frac{\partial f}{\partial y}$ parameter for the beta-plane approximation. The beta-plane approximation is obtained by expanding the Coriolis parameter f around a latitude ϕ_0 and retaining only the first two terms: $f = f_0 + \beta y$, where $y = R(\phi_0 + \phi')$ and f_0 is the standard Coriolis parameter.

(a) The surface water velocity is obtained by taking the limit $\gamma z = \lambda \to 0$. Therefore:

$$\lim_{\lambda\to 0} u = \lim_{\lambda\to 0} \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\lambda}(\cos\lambda + \sin\lambda) = \frac{\tau_w}{\rho\sqrt{2Kf}}(1+\lambda)(1+\lambda) = \frac{\tau_w}{\rho\sqrt{2Kf}}$$

$$\lim_{\lambda\to 0} v = \lim_{\lambda\to 0} \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\lambda}(\sin\lambda - \cos\lambda) = \frac{\tau_w}{\rho\sqrt{2Kf}}(1+\lambda)(\lambda-1) = -\frac{\tau_w}{\rho\sqrt{2Kf}}$$

The surface water velocity is at a 45 degrees angle with respect to the wind stress. Its x component is equal to:

$$\frac{\tau_w}{\rho\sqrt{2Kf}} = \frac{0.1}{1000\sqrt{2*0.001*9.94*10^{-4}}} = 0.2242\frac{\mathrm{m}}{\mathrm{s}}$$

(b) Since $\mathrm{e}^{\gamma\mathrm{z}}$ is always positive, the vector will be opposite when:

$$\cos\gamma z + \sin\gamma z = -1,$$

$$\sin\gamma z - \cos\gamma z = 1.$$

So:

$$2\sin\gamma z = 0,$$

$$\gamma z = n\pi, z < 0$$

$$z = \frac{-1\pi}{\gamma} = \frac{-\pi}{\gamma} = \frac{-3.1415}{\sqrt{\frac{9.94*10^{-4}}{0.001*2}}} = -14.0874\mathrm{m}$$

(c) Construct a following integral:

$$\int_{-\infty}^{0} \bar{u}\rho \mathrm{d}z,$$

$$\int_{-\infty}^{0} u\rho \mathrm{d}z = \rho\int_{-\infty}^{0} \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\gamma z}(\cos\gamma z + \sin\gamma z)\mathrm{d}z = \frac{\rho\tau_w}{\rho\sqrt{2Kf}}\frac{1}{\gamma}\int_{-\infty}^{0} e^{\lambda}(\cos\lambda + \sin\lambda)\mathrm{d}\lambda =$$

$$C\left(\int_{-\infty}^{0} e^{\lambda}\cos\lambda \mathrm{d}\lambda + \int_{-\infty}^{0} e^{\lambda}\sin\lambda \mathrm{d}\lambda\right) = C(0.5 - 0.5) = 0,$$

and for v:

$$\int_{-\infty}^{0} v\rho \mathrm{d}z = \rho \int_{-\infty}^{0} \frac{\tau_w}{\rho\sqrt{2Kf}} e^{\gamma z}(\sin\gamma z - \cos\gamma z)\mathrm{d}z = \frac{\rho\tau_w}{\rho\sqrt{2Kf}}\frac{1}{\gamma}\int_{-\infty}^{0} e^{\lambda}(\sin\lambda - \cos\lambda)\mathrm{d}\lambda =$$

$$C(\int_{-\infty}^{0} e^{\lambda}\sin\lambda \mathrm{d}\lambda - \int_{-\infty}^{0} e^{\lambda}\cos\lambda \mathrm{d}\lambda) = C(-0.5-0.5)) = -C = -\frac{\rho\tau_w}{\rho\sqrt{2Kf}}\frac{1}{\gamma} = -\frac{\tau_w}{\gamma\sqrt{2Kf}},$$

$$-\frac{\tau_w}{\gamma\sqrt{2Kf}} = -1005.38\frac{\mathrm{kg}}{\mathrm{m}}$$

the net mass transport is at 90 degree angle with respect to stress, in the direction of the negative y.

(d) To obtain the expression for β parameter, expand the Coriolis parameter around ϕ_0:

$$f = 2\Omega\sin(\phi_0 + \phi') \approx 2\Omega\sin(\phi_0) + \frac{2\Omega\cos(\phi_0)}{r_0} r_0(\phi_0 - \phi') = f_0 + \beta y$$

therefore:

$$\beta = \frac{2\Omega\cos(\phi_0)}{r_0}$$

in our case:

$$\beta = \frac{2\Omega\cos(\phi_0)}{r_0} = \frac{2 * 7.2921 * 10^{-5} * 0.7314}{6100000} = 1.7486 * 10^{-11}\frac{1}{\mathrm{s}\times\mathrm{m}}$$

Score: 1 point for a, 1 point for b, 2 points for c, 1 point for d.

3. Determine the ratio of the maximum possible regular anticyclonic gradient wind speed to the geostrophic wind speed for the same pressure gradient.

From the gradient wind equation we have:

$$U = -\frac{fR}{2} \pm \sqrt{\frac{f^2R^2}{4} + fRU_g}.$$

For the regular anticyclone we have $R < 0$ and we take the minus sign:

$$U = -\frac{fR}{2} - \sqrt{\frac{f^2R^2}{4} + fRU_g},$$

For U to have the maximum value, the term under the square root must be equal to zero, therefore:

$$\frac{f^2R^2}{4} + fRU_g = 0,$$

$$U_g = -\frac{fR}{4},$$

and:

$$U_{max} = -\frac{fR}{2}.$$

Therefore:

$$\frac{U_{max}}{U_g} = \frac{-\frac{fR}{2}}{-\frac{fR}{4}} = 2$$

Score: 1 point for gradient wind equation, 2 points for the geostrophic wind, 2 points the ratio.

4. Chain rule:

$$(\frac{\partial p}{\partial x})_z = (\frac{\partial p}{\partial x})_\theta - (\frac{\partial p}{\partial z})_x(\frac{\partial z}{\partial x})_\theta,$$

we use the hydrostatic balance:

$$(\frac{\partial p}{\partial x})_z = (\frac{\partial p}{\partial x})_\theta + \rho g(\frac{\partial z}{\partial x})_\theta,$$

divide by ρ and input the ideal gas law:

$$\frac{1}{\rho}(\frac{\partial p}{\partial x})_z = RT(\frac{\partial \ln p}{\partial x})_\theta + (\frac{\partial \phi}{\partial x})_\theta,$$

natural logarithm of potential temperature:

$$\ln \theta = \ln T - \frac{R}{c_p} \ln p + C,$$

the derivative while keeping the θ constant:

$$(\frac{\partial \ln p}{\partial x})_\theta = \frac{c_p}{R}(\frac{\partial \ln T}{\partial x})_\theta,$$

input to the final equation:

$$\frac{1}{\rho}(\frac{\partial p}{\partial x})_z = RT\frac{c_p}{R}(\frac{\partial \ln T}{\partial x})_\theta + (\frac{\partial \phi}{\partial x})_\theta = c_p(\frac{\partial T}{\partial x})_\theta + (\frac{\partial \phi}{\partial x})_\theta.$$

Score: 1 point for hydrostatic balance, 1 point for the geopotential, 1 point for ideal gas law, 2 points for the rest.