Exercise Sheet 6

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1. **Virtual mass of an accelerating body in ideal fluid.** For a rigid body moving with velocity $\mathbf{U}(t)$ through an ideal fluid, which was initially at rest, the velocity potential has the form

$$\phi(\mathbf{r}, t) = \mathbf{U}(t) \cdot \mathbf{\Phi}[\mathbf{r} - \mathbf{R}(t)], \tag{1}$$

where $\mathbf{\Phi}$ depends on the geometry of the immersed body and $\mathbf{R}(t)$ denotes the position of a reference point of the body.

(a) Use the appropriate form of the Bernoulli equation to show that the force on the body due to the pressure field,

$$\mathbf{F} = -\int_{\partial V} p\mathbf{n} ds \tag{2}$$

is of the form

$$F = \rho \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \cdot \int_{\partial V} \mathbf{\Phi n} ds + \rho \int_{\partial V} \left(\frac{1}{2}\mathbf{u}^2 - \mathbf{U} \cdot \mathbf{u} \right) \mathbf{n} ds \tag{3}$$

where ∂V is the boundary of the immersed object.

(b) Show that if $\mathrm{d}\mathbf{U}/\mathrm{d}t = 0$, then

$$\mathbf{F} \cdot \mathbf{U} = 0, \tag{4}$$

and interpret this result physically. Use the identity (valid for $\nabla^2\phi = 0$)

$$\frac{1}{2} \int_{\partial V} (\nabla\phi)^2 \mathbf{n} \mathrm{d}s = \int_{\partial V} \nabla\phi(\nabla\phi \cdot \mathbf{n}) \mathrm{d}s. \tag{5}$$

(c) The velocity potential $\phi(r,\theta)$ (in spherical coordinates) of the flow around a stationary sphere of radius a immersed in a uniform flow $\mathbf{U} = U\hat{\mathbf{z}}$ is

$$\phi(r,\theta) = Ur\cos\theta\left(1+\frac{1}{2}\frac{a^3}{r^3}\right). \tag{6}$$

Since the flow has axial symmetry, being $\hat{\mathbf{z}}$ the symmetry axis, the potential ϕ does not depend on the azimuth angle φ.

Show that, for the sphere moving in a fluid which was initially at rest,

$$\mathbf{\Phi}[\mathbf{r}-\mathbf{R}(t)] = -\frac{1}{2}a^3\frac{\mathbf{r}-\mathbf{R}(t)}{|\mathbf{r}-\mathbf{R}(t)|^3}. \tag{7}$$

If the sphere is accelerated with $\mathrm{d}\mathbf{U}/\mathrm{d}t = -(\mathrm{d}U/\mathrm{d}t)\hat{\mathbf{z}}$, then show that

$$\mathbf{F} = M'\frac{\mathrm{d}U}{\mathrm{d}t}\hat{\mathbf{z}}, \tag{8}$$

where

$$M' = \frac{2}{3}\pi a^3\rho \tag{9}$$

is the *virtual mass* of the sphere.

2. Obtain the two-dimensional irrotational flow past a circular cylinder in terms of the stream function ψ, which is related to the velocity field $\mathbf{u}$ through

$$\mathbf{u} = \nabla\times(\psi\hat{\mathbf{z}}), \tag{10}$$

where $\hat{\mathbf{z}}$ is the unit vector perpendicular to the plane of the flow.

3. **Force on a cylinder with circulation.** Consider a stationary circular cylinder immersed in a uniform flow $U\hat{\mathbf{x}}$. To simulate a rotating cylinder, we introduce a bidimensional free vortex

$$\mathbf{u} = \frac{k}{r}\mathbf{e}_\theta, \tag{11}$$

with a constant k, which measures the strength of the vortex.

   (a) Calculate the constant k in terms of the circulation Γ.

   (b) Find the velocity components of the flow resulting from superposition of velocity potentials.

   (c) Calculate the pressure field.

   (d) Calculate the force the fluid exerts on the cylinder.