Dynamics of the Atmosphere and the Ocean

Lecture 11

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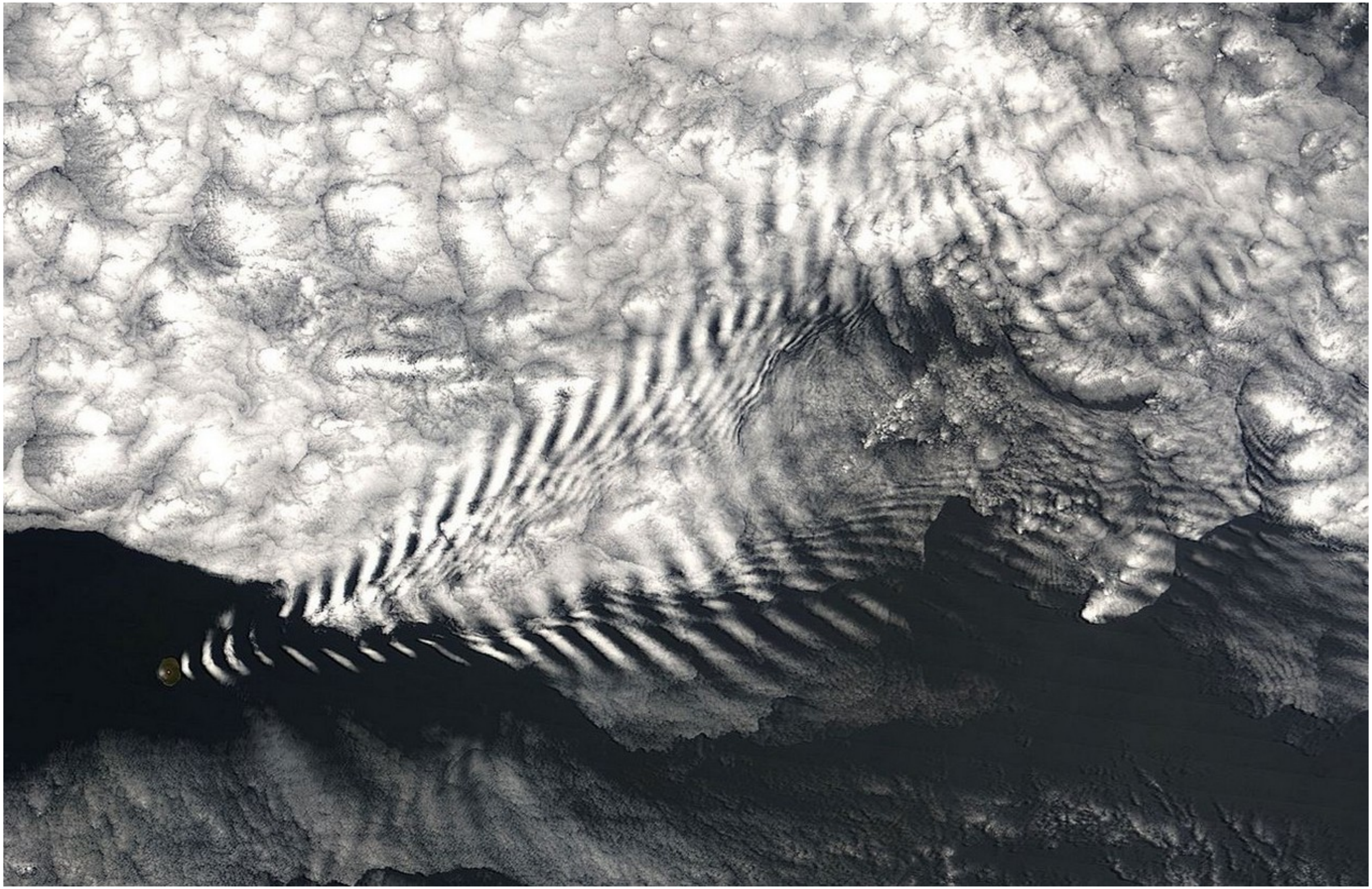
2022-2023 Fall



THE PERTURBATION METHOD

In the perturbation method, all field variables are divided into two parts, a basic state portion, which is usually assumed to be independent of time and position and a perturbation portion, which is the local deviation of the field from the basic state, e.g $u(x, t) = \overline{u} + u'(x, t)$

and

$$u \frac{\partial u}{\partial x} = \left(\overline{u} + u'\right) \frac{\partial}{\partial x} \left(\overline{u} + u'\right) = \overline{u} \frac{\partial u'}{\partial x} + u' \frac{\partial u'}{\partial x}$$

The basic assumptions of perturbation theory are that the basic state variables must themselves satisfy the governing equations when the perturbations are set to zero, and the perturbation fields must be small enough so that all terms in the governing equations that involve products of the perturbations can be neglected:

$$|u'/\overline{u}| \ll 1$$

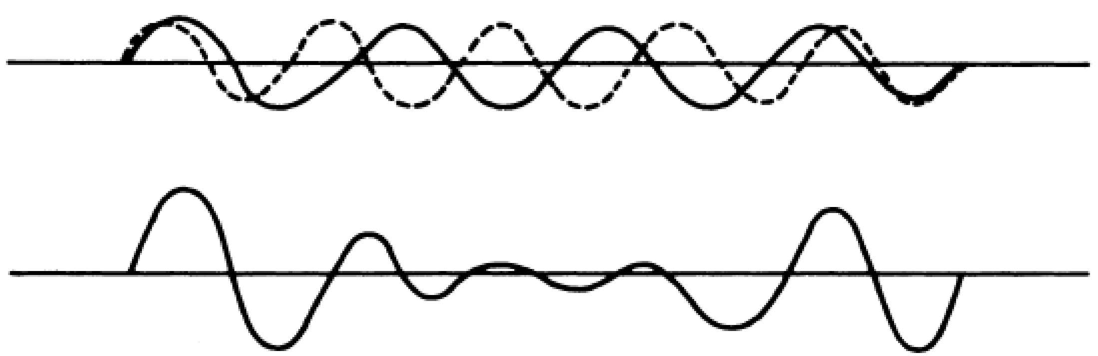
$$\left|\overline{u}\partial u'/\partial x\right| \gg \left|u'\partial u'/\partial x\right|$$

If terms that are products of the perturbation variables are neglected, the nonlinear governing equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients. These equations can then be solved by standard methods to determine the character and structure of the perturbations in terms of the known basic state.
For equations with constant coefficients the solutions are sinusoidal or exponential.

Dispersion and Group Velocity

For propagating waves, ν generally depends on the wave number of the perturbation as well as the physical properties of the medium. Thus, because $c = \nu/k$, the phase speed also depends on the wave number (dispersive waves) except in the special case where $\nu \sim k$ (non-dispersive waves).
The formula that relates ν and k is called a dispersion relationship.

Nondispersive waves (e.g. acoustic), have phase speeds that are independent of the wave number. A spatially localized disturbance consisting of a number of Fourier wave components (a wave group) will preserve its shape as it propagates in space at the phase speed of the wave.
For dispersive waves, however, the shape of a wave group will not remain constant as the group propagates. The individual Fourier components of a wave group may either reinforce or cancel each other. Furthermore, the group generally broadens in the course of time, that is, the energy is dispersed.



Wave groups formed from two sinusoidal components of slightly different wavelengths. For nondispersive waves, the pattern in the lower part of the diagram propagates without change of shape. For dispersive waves, the shape of the pattern changes in time.

When waves are dispersive, the speed of the wave group is generally different from the average phase speed of the individual Fourier components. Hence, as shown below, individual wave components may move either more rapidly or more slowly than the wave group as the group propagates along.

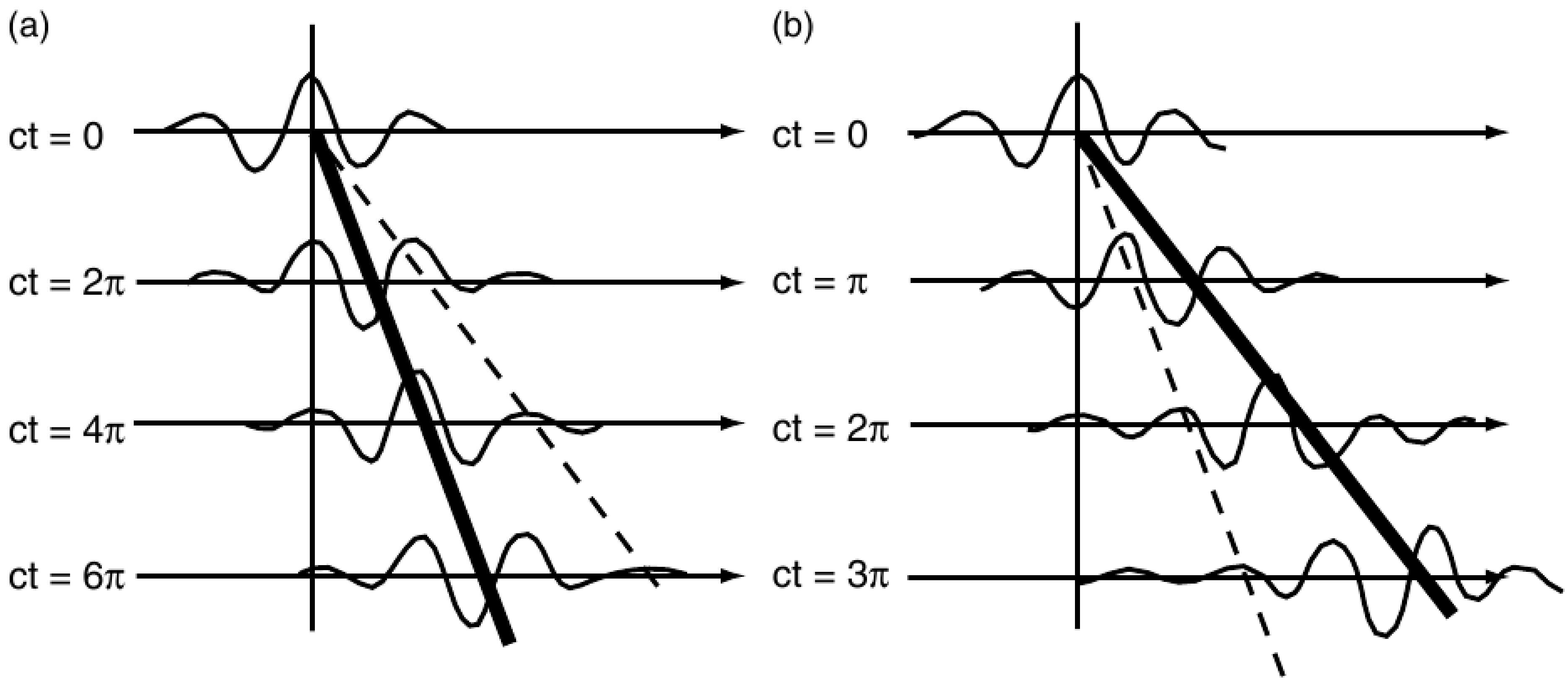


Fig. 7.4 Schematic showing propagation of wave groups: (a) group velocity less than phase speed and (b) group velocity greater than phase speed. Heavy lines show group velocity, and light lines show phase speed.

Group velocity (propagation velocity of an observable disturbance and hence the energy)

Consider the superposition of two horizontally propagating waves of equal amplitude but slightly different wavelengths with wave numbers and frequencies differing by $2\delta k$ and $2\delta \nu$. The total disturbance is thus:

$$\Psi\,(x,t) = \exp\{i\,[(k+\delta k)\,x - (\nu+\delta\nu)\,t]\} + \exp\{i\,[(k-\delta k)\,x - (\nu-\delta\nu)\,t]\}$$

In the above for brevity the Re[ ] is omitted, and it is understood that only the real part of the right-hand side has physical meaning.
Rearranging terms and applying the Euler formula gives:

$$\begin{aligned}\Psi &= \left[e^{i(\delta kx-\delta\nu t)} + e^{-i(\delta kx-\delta\nu t)}\right] e^{i(kx-\nu t)} \\ &= 2\cos(\delta kx - \delta\nu t)\, e^{i(kx-\nu t)}\end{aligned}$$

i.e. disturbance is the product of a high-frequency carrier of wavelength $2\pi/k$ of phase speed, ν/k being the average of the two Fourier components, and a low-frequency envelope of wavelength $2\pi/\delta k$ that travels at the speed $\delta\nu/\delta k$. Thus, in the limit as $\delta k \to 0$, the horizontal velocity of the envelope, or group velocity, is just:

$$c_{gx} = \partial\nu/\partial k$$

Acoustic or Sound Waves

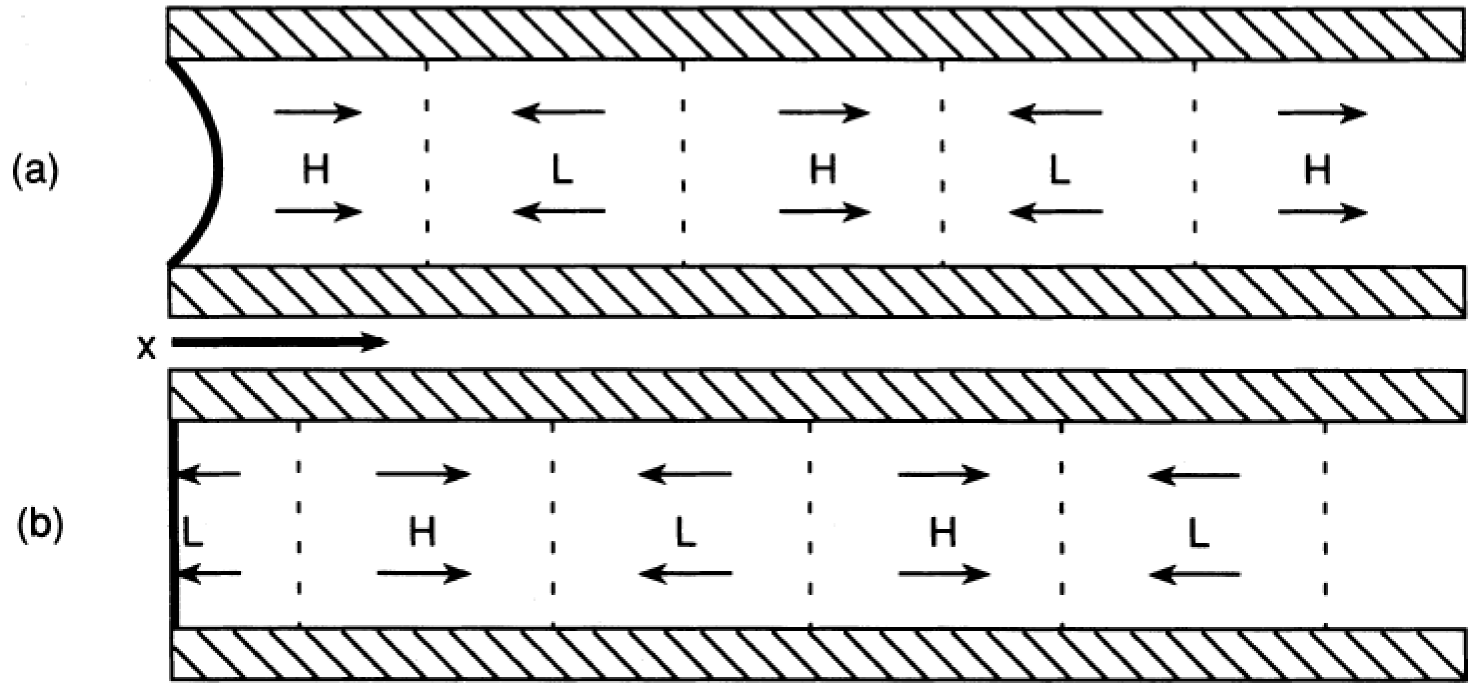


Fig. 7.5 Schematic diagram illustrating the propagation of a sound wave in a tube with a flexible diaphragm at the left end. Labels *H* and *L* designate centers of high and low perturbation pressure. Arrows show velocity perturbations. (b) The situation 1/4 period later than in (a) for propagation in the positive *x* direction.

Sound waves, or acoustic waves, are longitudinal waves. To introduce the perturbation method we consider the problem illustrated above.
We assume that $u = u(x,t)$.

The momentum equation, continuity equation, and thermodynamic energy equation for adiabatic motion are, respectively:

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \qquad D/Dt = \partial/\partial t + u\partial/\partial x$$

$$\frac{D \ln \theta}{Dt} = 0$$

Using $\theta = (p/\rho R)(p_s/p)^{R/c_p}$ we may combine last equations:

$$\frac{1}{\gamma}\frac{D \ln p}{Dt} - \frac{D \ln \rho}{Dt} = 0 \qquad \gamma = c_p/c_v$$

$$\frac{1}{\gamma}\frac{D \ln p}{Dt} + \frac{\partial u}{\partial x} = 0$$

Adopting perturbation theory and substituting to the first and last equations

$$\begin{aligned} u(x,t) &= \overline{u} + u'(x,t) \\ p(x,t) &= \overline{p} + p'(x,t) \\ \rho(x,t) &= \overline{\rho} + \rho'(x,t) \end{aligned}$$

After substitution one obtains:

$$\frac{\partial}{\partial t}\left(\overline{u}+u'\right)+\left(\overline{u}+u'\right)\frac{\partial}{\partial x}\left(\overline{u}+u'\right)+\frac{1}{\left(\overline{\rho}+\rho'\right)}\frac{\partial}{\partial x}\left(\overline{p}+p'\right)\;=\;0$$

$$\frac{\partial}{\partial t}\left(\overline{p}+p'\right)+\left(\overline{u}+u'\right)\frac{\partial}{\partial x}\left(\overline{p}+p'\right)+\gamma\left(\overline{p}+p'\right)\frac{\partial}{\partial x}\left(\overline{u}+u'\right)\;=\;0$$

We next observe that provided $\left|\rho'/\overline{\rho}\right| \ll 1$ we can use the binomial expansion to approximate the density term as:

$$\frac{1}{\left(\overline{\rho}+\rho'\right)}=\frac{1}{\overline{\rho}}\left(1+\frac{\rho'}{\overline{\rho}}\right)^{-1}\approx\frac{1}{\overline{\rho}}\left(1-\frac{\rho'}{\overline{\rho}}\right)$$

Neglecting products of the perturbation quantities and noting that the basic state fields are constants, we obtain the linear perturbation equations:

$\partial/\partial x$ | $$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)u'+\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x}=0$$

minus

$(\partial/\partial t+\overline{u}\partial/\partial x)$ | $$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)p'+\gamma\,\overline{p}\frac{\partial u'}{\partial x}=0$$

we get the standard wave equation:

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)^2 p' - \frac{\gamma\overline{p}}{\overline{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0$$

A simple solution representing a plane sinusoidal wave propagating in x is

$$p' = A\exp\left[ik\left(x - ct\right)\right]$$

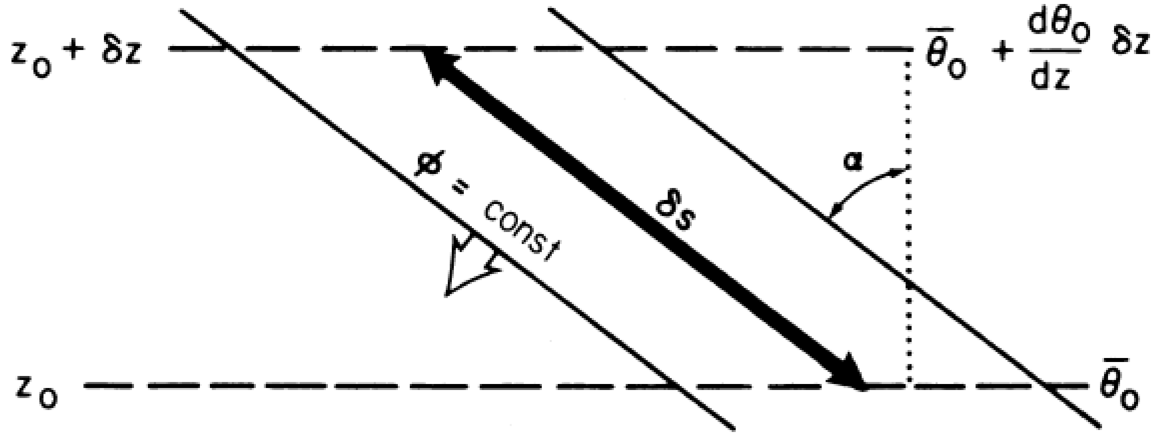
where for brevity we omit the Re{ } notation. Substituting the assumed solution we find that the phase speed c must satisfy:

$$(-ikc + ik\overline{u})^2 - \left(\gamma\overline{p}/\overline{\rho}\right)(ik)^2 = 0$$

$$c = \overline{u} \pm (\gamma\overline{p}/\overline{\rho})^{1/2} = \overline{u} \pm \left(\gamma R\overline{T}\right)^{1/2}$$

where $c_s \equiv \left(\gamma R\overline{T}\right)^{1/2}$ is the adiabatic speed of sound.

Pure Internal Gravity Waves



$$\delta z = \delta s \cos\alpha$$

Fig. 7.8 Parcel oscillation path (heavy arrow) for pure gravity waves with phase lines tilted at an angle α to the vertical.

The vertical buoyancy force per unit mass is $-N^2\delta z$ and the component of the buoyancy force parallel to the tilted path is: $-N^2\delta z\cos\alpha = -N^2(\delta s\cos\alpha)\cos\alpha = -(N\cos\alpha)^2\delta s$

The momentum equation for the parcel oscillation is then

$$\frac{d^2(\delta s)}{dt^2} = -(N\cos\alpha)^2\delta s$$

with the general solution $\delta s = \exp[\pm i(N\cos\alpha)t]$ - oscillations of frequency $\nu = N\cos\alpha$.

The above heuristic derivation can be verified by considering the linearized equations for two-dimensional internal gravity waves. For simplicity, we employ the Boussinesq approximation. Neglecting effects of rotation, the basic equations for two-dimensional motion of an incompressible atmosphere may be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0$$

$$\theta = \frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{\kappa}$$

$$\ln\theta = \gamma^{-1}\ln p - \ln\rho + \text{constant}$$

We now apply perturbations and linearize the above.

$$\rho = \rho_0 + \rho' \qquad u = \overline{u} + u'$$

$$p = \overline{p}(z) + p' \qquad w = w'$$

$$\theta = \overline{\theta}(z) + \theta'$$

The basic state zonal flow u and the density ρ_0 are both assumed to be constant. The basic state pressure field must satisfy the hydrostatic equation $d\overline{p}/dz = -\rho_0 g$ while basic state of temperature satisfies $\ln\overline{\theta} = \gamma^{-1}\ln\overline{p} - \ln\rho_0 + \text{constant}$

The linearized equations are obtained by substituting from the above into equations of motion and neglecting all terms that are products of the perturbation variables. Thus, for example, the last two terms in the vertical component of momentum equation are approximated as

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = \frac{1}{\rho_0 + \rho'}\left(\frac{d\overline{p}}{dz} + \frac{\partial p'}{\partial z}\right) + g$$

$$\approx \frac{1}{\rho_0}\frac{d\overline{p}}{dz}\left(1 - \frac{\rho'}{\rho_0}\right) + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g = \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{\rho'}{\rho_0}g$$

The perturbed form of energy equation is obtained by noting that

$$\ln\left[\overline{\theta}\left(1 + \frac{\theta'}{\overline{\theta}}\right)\right] = \gamma^{-1}\ln\left[\overline{p}\left(1 + \frac{p'}{\overline{p}}\right)\right] - \ln\left[\rho_0\left(1 + \frac{\rho'}{\rho_0}\right)\right] + \text{const.}$$

recalling that $\ln(ab) = \ln(a) + \ln(b)$ and that $\ln(1+\varepsilon) \approx \varepsilon$ for any $\varepsilon \ll 1$

one gets

$$\frac{\theta'}{\overline{\theta}} \approx \frac{1}{\gamma}\frac{p'}{\overline{p}} - \frac{\rho'}{\rho_0}$$

speed of sound squared

$$\rho' \approx -\rho_0 \frac{\theta'}{\overline{\theta}} + \frac{p'}{c_s^2} \qquad c_s^2 \equiv \overline{p}\gamma/\rho_0$$

For buoyancy wave density fluctuations due to pressure changes are small compared with those due to temperature changes: $\left|\rho_0 \theta'/\overline{\theta}\right| \gg \left|p'/c_s^2\right|$

Therefore, to a first approximation $\theta'/\overline{\theta} = -\rho'/\rho_0$ and the linearized equations of motion are:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)u' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0$$

∂( )/∂z

minus

∂( )/∂x

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)w' + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}}g = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\theta' + w'\frac{d\overline{\theta}}{dz} = 0$$

Operations on first two equations give the y component of the vorticity equation:

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{g}{\overline{\theta}}\frac{\partial \theta'}{\partial x} = 0$$

while eliminating θ' and u' from the two last equation gives

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

$$N^2 \equiv g\, d \ln \overline{\theta} / dz$$

Brunt-Vaisala frequency is assumed to be constant.
Harmonic wave solutions have the form:

$$w' = \mathrm{Re}\left[\hat{w} \exp(i\phi)\right] = w_r \cos\phi - w_i \sin\phi$$

$$\hat{w} = w_r + i w_i \qquad \phi = kx + mz - \nu t$$

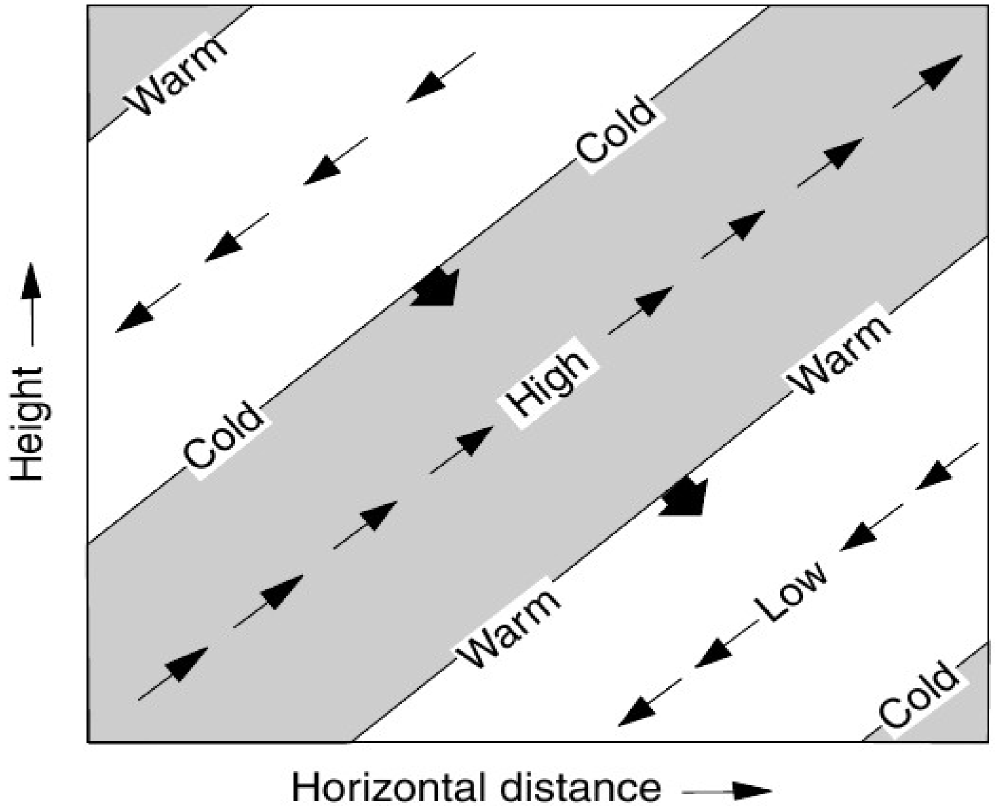
Here the horizontal wave number k is real because the solution is always sinusoidal in x. The vertical wave number $m = m_r + im_i$ may. be complex or negative. When m is real, the total wave number may be regarded as a vector $\kappa \equiv (k, m)$, directed perpendicular to lines of constant phase, and in the direction of phase increase, whose components, $k = 2\pi/Lx$ and $m = 2\pi/Lz$, are inversely proportional to the horizontal and vertical wavelengths, respectively.

Substitution of the assumed solution yields the dispersion relationship

$$(\nu - \overline{u}k)^2 \left(k^2 + m^2\right) - N^2 k^2 = 0$$

$$\hat{\nu} \equiv \nu - \overline{u}k = \pm Nk / \left(k^2 + m^2\right)^{1/2} = \pm Nk / |\boldsymbol{\kappa}|$$

$\hat{\nu}$, the *intrinsic frequency*, is the frequency relative to the mean wind.



If we let $k > 0$ and $m < 0$, then lines of constant phase tilt eastward with increasing height (i.e., for $\varphi = kx + mz$ to remain constant as x increases, z must also increase when $k > 0$ and $m < 0$). The choice of the positive root in then corresponds to eastward and downward phase propagation relative to the mean flow with horizontal and vertical phase speeds $c_x = \tilde{\nu}/k$ and , $c_z = \tilde{\nu}/m$.

Idealized cross section showing phases of pressure, temperature, and velocity perturbations for an internal gravity wave. Thin arrows indicate the perturbation velocity field, blunt solid arrows the phase velocity. Shading shows regions of upward motion.

The components of the group velocity, c_{gx} and c_{gz} , are given by

$$c_{gx} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}}$$

$$c_{gz} = \frac{\partial \nu}{\partial m} = \pm \frac{(-Nkm)}{\left(k^2 + m^2\right)^{3/2}}$$

The vertical component of group velocity has a sign opposite to that of the vertical phase speed relative to the mean flow (downward phase propagation implies
upward energy propagation). Furthermore, the group velocity vector is parallel to lines of constant phase.
Internal gravity waves have the remarkable property that group velocity is perpendicular to the direction of phase propagation. Because energy propagates at the group velocity this implies that energy propagates parallel to the wave crests and troughs, rather than perpendicular to them as in acoustic waves or shallow water gravity waves.

Topographic Waves

When air with mean wind speed u is forced to flow over a sinusoidal pattern of ridges under statically stable conditions, individual air parcels are alternately dis placed upward and downward from their equilibrium levels and will thus undergo buoyancy oscillations as they move across the ridges as shown.

In this case there are solutions in the form of waves that are stationary relative to the ground, i.e, $v = 0$. For such stationary waves, w depends only on (x, z) and solution simplifies to:

$$\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + \frac{N^2}{\overline{u}^2} w' = 0$$

resulting in dispersion relationship:

$$m^2 = N^2 \Big/ \overline{u}^2 - k^2$$

which determines vertical structure of such waves.

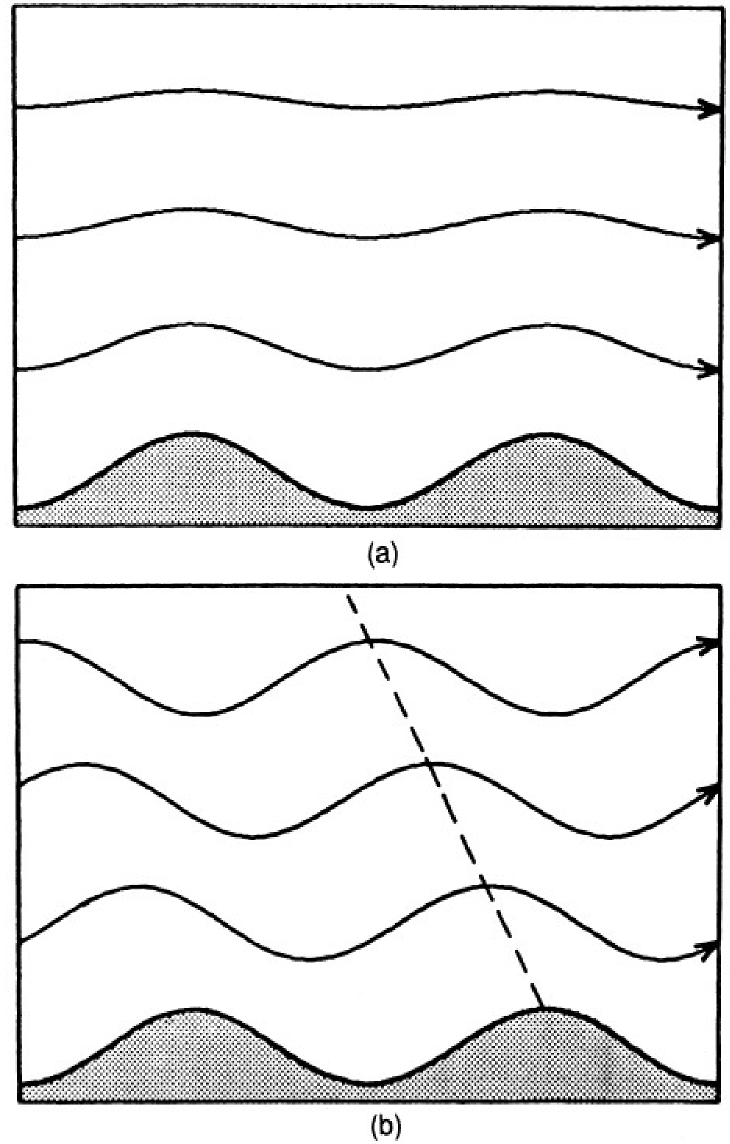


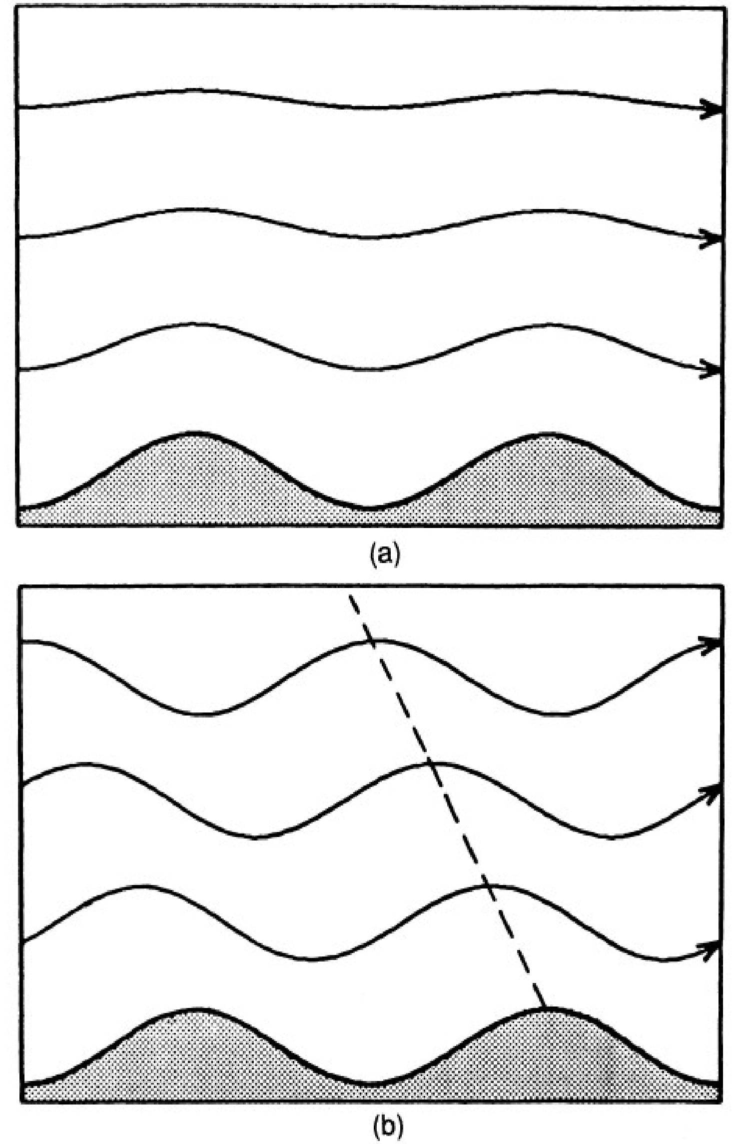
Fig. 7.10 Streamlines in steady flow over an infinite series of sinusoidal ridges for the narrow rid case (a) and broad ridge case (b). The dashed line in (b) shows the phase of maximu upward displacement. (After Durran, 1990.)

For $|\overline{u}| < N/k$, m is real. Solutions have the form of vertically propagating waves:

$$w' = \hat{w} \exp[i(kx + mz)]$$

When $m^2 < 0$, $m = im_i$ is imaginary and the solution will have the form of vertically trapped waves:

$$w' = \hat{w} \exp(ikx) \exp(-m_i z)$$



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Pure Inertial Oscillations

If the basic state flow is assumed to be a zonally directed geostrophic wind u_g, and it is assumed that the parcel displacement does not perturb the pressure field, the approximate equations of motion become:

$$\frac{Du}{Dt} = fv = f\frac{Dy}{Dt}$$

$$\frac{Dv}{Dt} = f\left(u_g - u\right)$$

We consider a parcel that is moving with the geostrophic basic state motion at a position $y = y0$. If the parcel is displaced across stream by a distance δy, we can obtain its new zonal velocity from the integrated form the first equation:

$$u\left(y_0 + \delta y\right) = u_g\left(y_0\right) + f\delta y$$

The geostrophic wind at $y0 + \delta y$ can be approximated as

$$u_g\left(y_0 + \delta y\right) = u_g\left(y_0\right) + \frac{\partial u_g}{\partial y}\delta y$$

Substituting to the second equation of motion gives:

$$\frac{Dv}{Dt} = \frac{D^2 \delta y}{Dt^2} = -f\left(f - \frac{\partial u_g}{\partial y}\right)\delta y = -f\frac{\partial M}{\partial y}\delta y$$ the *absolute momentum*, $M \equiv fy - u_g$.

Depending on the sign of the coefficient on the right-hand side, the parcel will either be forced to return to its original position or will accelerate further from that position. This determines the condition for inertial instability:

$$f\frac{\partial M}{\partial y} = f\left(f - \frac{\partial u_g}{\partial y}\right)\begin{cases} > 0 & \text{stable} \\ = 0 & \text{neutral} \\ < 0 & \text{unstable} \end{cases}$$

Viewed in an inertial reference frame, instability results from an imbalance between the pressure gradient and inertial forces for a parcel displaced radially in an axisymmetric vortex.

Inertia–Gravity Waves

When the flow is both inertially and gravitationally stable, parcel displacements are resisted by both rotation and buoyancy. The resulting oscillations are called inertia–gravity waves. The dispersion relation for such waves can be analyzed using a variant of the parcel method. Consider parcel oscillations along a slantwise path in the (y, z) plane:

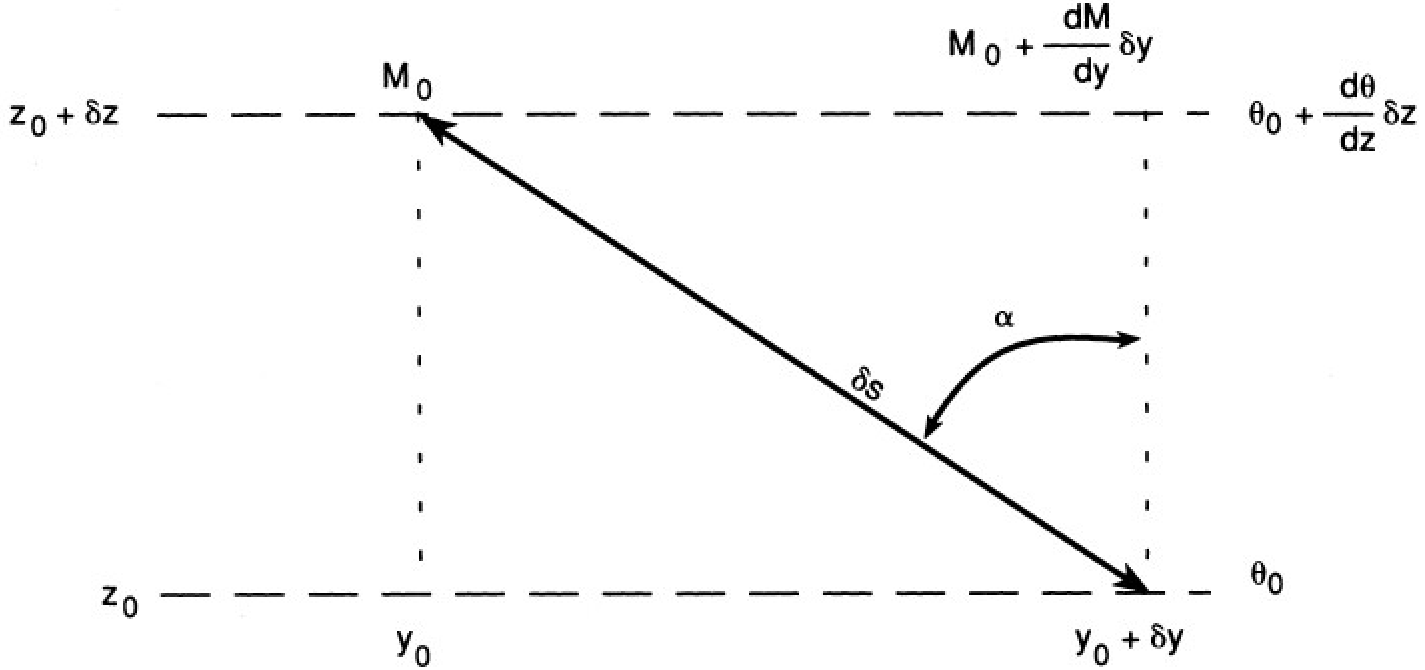

Fig. 7.11 Parcel oscillation path in meridional plane for an inertia–gravity wave. See text for definition of symbols.

For a vertical displacement δz the buoyancy force component parallel to the slope of the parcel oscillation is $-N^2\delta z \cos\alpha$, and for a meridional displacement δy the Coriolis (inertial) force component parallel to the slope of the parcel path is $-f^2\delta y \sin\alpha$, where we have assumed that the geostrophic basic flow is constant in latitude.

Thus, the harmonic oscillator equation for the parcel is modified to the form:

$$\frac{D^2\delta s}{Dt^2} = -(f\sin\alpha)^2\,\delta s - (N\cos\alpha)^2\,\delta s$$

The frequency satisfies the dispersion relationship:

$$\nu^2 = N^2\cos^2\alpha + f^2\sin^2\alpha$$

In general $N^2 > f^2$ indicates that inertia–gravity wave frequencies must lie in the range $f \le |\nu| \le N$. The frequency approaches N as the trajectory slope approaches the vertical, and approaches f as the trajectory slope approaches the horizontal. For typical midlatitude tropospheric conditions, inertia–gravity wave periods are in the range of 10 min to 15 h.

The heuristic parcel derivation can again be verified by using the linearized dynamical equations. In this case, including rotation.

$$\frac{\partial u'}{\partial t} - fv' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} = 0$$

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}}g = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + w'\frac{d\overline{\theta}}{dz} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial z}\right) + N^2 w' = 0$$

ADJUSTMENT TO GEOSTROPHIC BALANCE

In the course of the lecture we showed that synoptic-scale motions in midlatitudes are in approximate geostrophic balance. Departures from this balance can lead to the excitation of inertia–gravity waves, which act to adjust the mass and momentum.

In order to discuss this adjustment we utilize the prototype shallow water system. For linearized disturbances about a basic state of no motion with a constant Coriolis parameter, f_0, the horizontal momentum and continuity equations are:

plus

$\partial(\ )/\partial x$

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$\partial(\ )/\partial y$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

Adopting the above calculation yields:

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$

$$c^2 \equiv gH \text{ and } \zeta' = \partial v'/\partial x - \partial u'/\partial y.$$

For $f_0 = 0$ (nonrotating system) the vorticity and height perturbations are uncoupled, and we get a two-dimensional shallow water wave equation for h':

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) = 0$$

with the following solution:

$$h' = A\exp[i(kx + ly - \nu t)]$$
$$\nu^2 = c^2\left(k^2 + l^2\right) = gH\left(k^2 + l^2\right).$$

However, for $f_0 \neq 0$ the h and ζ fields are coupled and for motions with time scales longer than $1/f_0$ (which is certainly true for synoptic-scale motions), the ratio of the first two terms

is given by

$$\frac{\left|\partial^2 h'/\partial t^2\right|}{\left|c^2\left(\partial^2 h'/\partial x^2 + \partial^2 h'/\partial y^2\right)\right|} \lesssim \frac{f_0^2 L^2}{gH}$$

which is small for $L \sim 1000$ km, provided that $H > 1$ km. Under such circumstances the time derivative term is small compared to the other two terms, and the equation states simply that the vorticity is in geostrophic balance.

If the flow is initially unbalanced, the complete equation can be used to describe the approach toward geostrophic balance provided that we can obtain a second relationship between h' and ζ' . Taking:

minus

$\partial(\ )/\partial y$
$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$\partial(\ )/\partial x$
$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

yields:

$$\frac{\partial \zeta'}{\partial t} + f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

This, combined with
$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

results in
$$\frac{\partial \zeta'}{\partial t} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0$$

the linearized potential vorticity conservation law.
Thus, letting Q' designate the perturbation potential vorticity, we obtain the conservation relationship:

$$Q'(x, y, t) = \zeta'/f_0 - h'/H = \text{Const.}$$

$$Q'(x, y, t) = Q'(x, y, 0)$$

This problem, solved by Rossby in the 1930s, is often referred to as the Rossby adjustment problem. As a simplified example of the adjustment process, we consider an idealized shallow water (ocean) system on a rotating plane with initial conditions:

$$u', v' = 0; \qquad h' = -h_0 \mathrm{sgn}(x)$$

Motionless “step” on the water surface. Using conservation relation

$$\left(\zeta'/f_0\right) - \left(h'/H\right) = \left(h_0/H\right) \mathrm{sgn}(x)$$

and eliminating ζ' yields

$$\frac{\partial^2 h'}{\partial t^2} - c^2 \left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2} \right) + f_0^2 h' = -f_0^2 h_0 \mathrm{sgn}(x)$$

which in the homogeneous case ($h_0 = 0$) yields the dispersion relation

$$\nu^2 = f_0^2 + c^2 \left(k^2 + l^2\right) = f_0^2 + gH \left(k^2 + l^2\right)$$

Because initially h' is independent of y, it will remain so for all time. Thus, in the final steady state solution becomes

$$-c^2 \frac{d^2 h'}{dx^2} + f_0^2 h' = -f_0^2 h_0 \, \mathrm{sgn}(x)$$

$$\frac{h'}{h_0} = \left\{ \begin{array}{ll} -1 + \exp(-x/\lambda_R) & \text{for } x > 0 \\ +1 - \exp(+x/\lambda_R) & \text{for } x < 0 \end{array} \right\}$$

$\lambda_R \equiv f_0^{-1} \sqrt{gH}$ is the Rossby *radius of deformation.*

The Rossby radius of deformation may be interpreted as the horizontal length scale over which the height field adjusts during the approach to geostrophic equilibrium.
For $|x| >> \lambda_R$ the original h' remains unchanged. Substituting from the last equation into

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

Shows that the steady velocity field is geostrophic and nondivergent:

$$u' = 0, \quad \text{and} \quad v' = \frac{g}{f_0}\frac{\partial h'}{\partial x} = -\frac{g h_0}{f_0 \lambda_R}\exp\left(-|x|/\lambda_R\right)$$

The steady-state solution is shown in the next page.

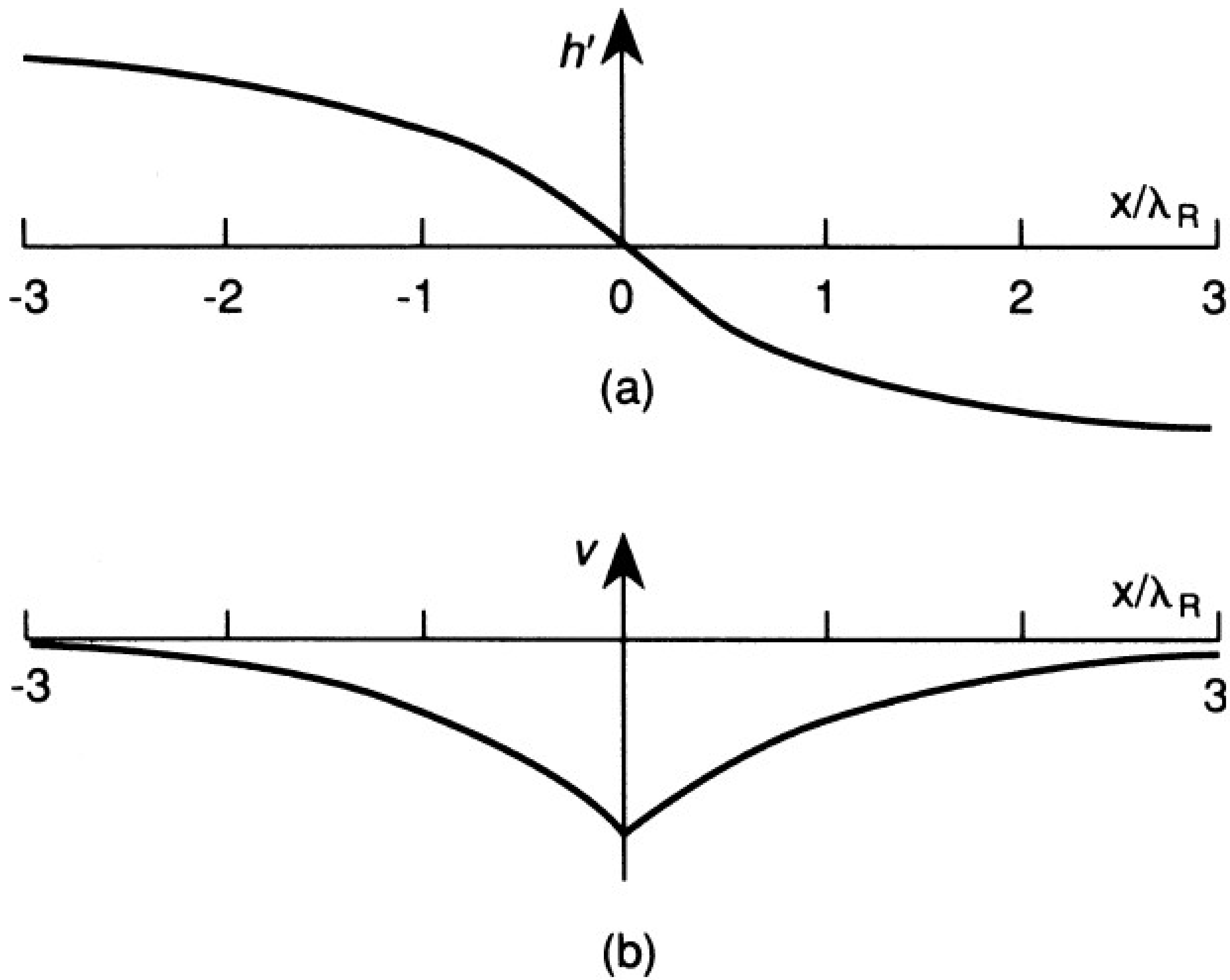

Fig. 7.13 The geostrophic equilibrium solution corresponding to adjustment from the initial state defined in (7.78). (a) Final surface elevation profiles; (b) the geostrophic velocity profile in the final state. (After Gill, 1982.)

Substituting:

$$\left(u', v', w', p'/\rho_0\right) = \mathrm{Re}\left[\left(\hat{u}, \hat{v}, \hat{w}, \hat{p}\right) \exp i\,(kx + ly + mz - \nu t)\right]$$

to the remaining four equation one obtains:

$$\hat{u} = \left(\nu^2 - f^2\right)^{-1} (\nu k + ilf\,)\,\hat{p}$$

$$\hat{v} = \left(\nu^2 - f^2\right)^{-1} (\nu l - ikf)\,\hat{p}$$

$$\hat{w} = -\left(\nu m / N^2\right)\hat{p}$$

which yields the dispersion relation for hydrostatic waves:

$$\nu^2 = f^2 + N^2\left(k^2 + l^2\right) m^{-2}$$

Because hydrostatic waves must have $(k^2 + l^2)/m^2 << 1$, the above indicates that for vertical propagation to be possible (m real) the frequency must satisfy the inequality $|f\,| < |\nu| << N$. I is just the limit of $\nu^2 = N^2\cos^2\alpha + f^2\sin^2\alpha$
when we let

$$\sin^2\alpha \rightarrow 1, \cos^2\alpha = \left(k^2 + l^2\right)/m^2$$

which is consistent with the hydrostatic approximation.
If axes are chosen to make $l = 0$, it may be shown that the ratio of the vertical to horizontal components of group velocity is given by

$$\left|c_{gz}/c_{gx}\right| = |k/m| = \left(\nu^2 - f^2\right)^{1/2}/N$$