Solutions for extra exercises for midterm exam

1. Combine the hydrostatic balance and ideal gas law to get:

$$\mathrm{d}z = -\frac{\mathrm{d}p}{p}\frac{RT}{g}$$

Integrate it from z_1 to z_2, and substitute $\langle T \rangle = \frac{\int_{p_1}^{p_2} T \mathrm{d}\ln p}{\int_{p_1}^{p_2} \mathrm{d}\ln p}$.

Thermal wind: we apply the derivative with respect to pressure to the geostrophic balance. Then we note that $\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$. Then we use the gas law:

$$\frac{\partial \bar{u}_g}{\partial p} = -\frac{R}{fp}\hat{k} \times \nabla_p T,$$

and then we multiply by $f\times$ to get:

$$\bar{f} \times \frac{\partial \bar{u}_g}{\partial p} = \frac{R}{p}\nabla_p T$$

Isothermal: using the chain rule:

$$(\frac{\partial p}{\partial x})_T = (\frac{\partial p}{\partial x})_z + (\frac{\partial p}{\partial z})_x(\frac{\partial z}{\partial x})_T,$$

$$(\frac{\partial p}{\partial x})_T = (\frac{\partial p}{\partial x})_z - \rho g\frac{\partial z}{\partial x})_T$$

$$(\frac{\partial p}{\partial x})_z = (\frac{\partial p}{\partial x})_T + \rho\frac{\partial \phi}{\partial x})_T,$$

divide by $\rho = RT/p$:

$$\frac{1}{\rho}(\frac{\partial p}{\partial x})_z = \frac{RT}{p}(\frac{\partial p}{\partial x})_T + (\frac{\partial \phi}{\partial x})_T,$$

$$\frac{1}{\rho}(\frac{\partial p}{\partial x})_z = RT(\frac{\partial \ln p}{\partial x})_T + (\frac{\partial \phi}{\partial x})_T,$$

2. The geopotential is:

$$\Phi(h) = \int_0^h g\mathrm{d}z.$$

For hydrostatic balance:

$$g = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{p}{RT}\frac{\mathrm{d}p}{\mathrm{d}z},$$

therefore

$$\Phi(h) = \int_0^h -\frac{p}{RT}\frac{\mathrm{d}p}{\mathrm{d}z}\mathrm{d}z = \int_{p_0}^{p(h)} -\frac{p}{RT}\mathrm{d}p,$$

assuming constant temperature:

$$\Phi(h) = -\frac{1}{RT}\int_{p_0}^{p(h)} p\mathrm{d}p = -\frac{p(h)^2 - p_0^2}{2RT}.$$

For the constant potential temperature:

$$\Phi(h) = \int_{p_0}^{p(h)} -\frac{p}{RT}\mathrm{d}p = \int_{p_0}^{p(h)} -\frac{p}{R\theta(\frac{p}{p_0})^{R/c_p}}\mathrm{d}p = -\frac{p_0^{R/c_p}}{R\theta}\int_{p_0}^{p(h)} p^{1-R/c_p}\mathrm{d}p =$$

$$= -\frac{p_0^{R/c_p}}{R\theta}\frac{1}{(2-R/c_p)}(p(h)^{2-R/c_p} - p_0^{2-R/c_p}).$$

It probably can be prettier.

3. Some of the exercises were done in the class

The integrals are:

$$U = \int_0^\infty (u - u_g)\mathrm{d}z = \frac{1}{\gamma}\int_0^\infty -u_g[e^{-\lambda}\cos(\lambda)]\mathrm{d}\lambda = \frac{-u_g}{\gamma}\int_0^\infty [e^{-\lambda}\cos(\lambda)]\mathrm{d}\lambda = \frac{-u_g}{2\gamma}$$

$$V = \int_0^\infty (v - v_g)\,\mathrm{d}z = \frac{1}{\gamma}\int_0^\infty u_g[e^{-\lambda}\sin(\lambda)]\,\mathrm{d}\lambda = \frac{u_g}{\gamma}\int_0^\infty [e^{-\lambda}\sin(\lambda)]\,\mathrm{d}\lambda = \frac{u_g}{2\gamma}$$

Therefore our vector is $\frac{u_g}{2\gamma}(-1, 1)$. It is oriented at an angle of 135 degrees, whereas the friction is oriented at an angle of 225 degrees. The difference is 90 degrees.

4. To derive the formula, we need to transform the first formula to isobaric coordinates:

$$\frac{u^2 \tan\varphi}{a} = \omega_r^2 a \sin\varphi \cos\varphi = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial \phi}{\partial y},$$

now use the relation:

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p},$$

and calculate the derivative with respect to p:

$$\frac{\partial \omega_r^2}{\partial p} = \frac{1}{a \sin\varphi \cos\varphi}\frac{\partial}{\partial y}(\frac{RT}{p}),$$

then integrate with respect to p:

$$\omega_r^2(p_2) - \omega_r^2(p_1) = \frac{R \ln(p_2/p_1)}{a \sin\varphi \cos\varphi}\frac{\partial \langle T \rangle}{\partial y}.$$

To calculate the temperature difference, we need to assume that the RHS is a function of p only:

$$\omega_r^2 = -\frac{1}{a \sin\varphi \cos\varphi}\frac{\partial \phi}{\partial y} = F(p),$$

or

$$\omega_r^2(p_2) - \omega_r^2(p_1) = \frac{R \ln(p_2/p_1)}{a \sin\varphi \cos\varphi}\frac{\partial \langle T \rangle}{\partial y} = G(p),$$

so

$$\frac{\partial \langle T \rangle}{\partial y} = C \sin\varphi \cos\varphi,$$

in spherical Earth coordinates:

$$\frac{\partial}{\partial y} = \frac{1}{a}\frac{\partial}{\partial \varphi},$$

therefore:

$$\frac{\partial \langle T \rangle}{\partial \varphi} = C\text{sin}\varphi\text{cos}\varphi,$$

$$\partial \langle T \rangle = \partial \varphi C\text{sin}\varphi\text{cos}\varphi,$$

$$T_{eq} - T_{po} = \Delta T = -\frac{C}{2}\text{cos}^2\varphi|_{90}^{0} = -\frac{C}{2},$$

therefore:

$$C = -2\Delta T,$$

$$\frac{\partial \langle T \rangle}{\partial \varphi} = -2\Delta T\text{sin}\varphi\text{cos}\varphi,$$

$$\omega_r^2(p_2) - \omega_r^2(p_1) = \frac{R\text{ln}(p_2/p_1)}{a^2\text{sin}\varphi\text{cos}\varphi}(-2\Delta T\text{sin}\varphi\text{cos}\varphi) = \frac{-2\Delta T R\text{ln}(p_2/p_1)}{a^2},$$

$$\Delta T = \frac{(\omega_r^2(p_2) - \omega_r^2(p_1))a^2}{-2R\text{ln}(p_2/p_1)} = \frac{v_r^2(p_2) - v_r^2(p_1)}{-2R\text{ln}(p_2/p_1)} = \frac{100^2}{(-2)(187)(\text{ln}(\frac{0.29}{9.5}))} = 7.6631\text{K}$$