# Turbulence and atmospheric boundary layer
# Lecture 3 and 4

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# Summary of lecture 2

1. Reynolds-stress transport equation
   - Subtract Reynolds equation for $\overline{u_i}$ from Navier-Stokes equation for $u_i$
   - In this way equation for $u_i'$ is obtained
   - Derive equation for $u_j'$ in analogous way
   - Multiply equation for $u_i'$ by $u_j'$, multiply equation for $u_j'$ by $u_i'$
   - Add both equations
   - Average resulting equation

2. Physical meaning of different terms in RS transport equation

# Reynolds-stress transport equations

## RS transport equation

$$\underbrace{\frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u_k} \frac{\partial \overline{u_i' u_j'}}{\partial x_k}}_{(1)} + \underbrace{\frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{1}{\rho}\left(\frac{\partial \overline{p' u_j'}}{\partial x_i} + \frac{\partial \overline{p' u_i'}}{\partial x_j}\right)}_{(3)} = \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\right)}}_{(4)}$$

$$- \underbrace{\overline{u_i' u_k'} \frac{\partial \overline{u_j}}{\partial x_k} - \overline{u_j' u_k'} \frac{\partial \overline{u_i}}{\partial x_k}}_{(5)} + \nu \underbrace{\frac{\partial^2 \overline{u_i' u_j'}}{\partial x_k \partial x_k}}_{(6)} - \underbrace{2\nu \overline{\left(\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{(7)}$$

$$+ \underbrace{\delta_{i3}\overline{b' u_j'} + \delta_{j3}\overline{b' u_i'}}_{(8)} + \underbrace{\epsilon_{ik3} f \overline{u_k' u_j'} + \epsilon_{jk3} f \overline{u_k' u_i'}}_{(9)}$$

# Reynolds-stress transport equations

1. Transport with mean velocity
2. Turbulent transport
3. Pressure transport
4. Pressure-strain rate tensor (redistribution)
5. Shear production term
6. Viscous transport
7. Dissipation tensor
8. Buoyancy production (positive or negative)
9. Coriolis term (redistribution)

# Transport equation for $\overline{u'^2}$

Take $i = j = 1$ in RS equation, $\overline{u_1' u_1'} = \overline{u'^2}$

## RS transport equation

$$\underbrace{\frac{\partial \overline{u_1'^2}}{\partial t} + \overline{u_k} \frac{\partial \overline{u_1'^2}}{\partial x_k}}_{(1)} + \underbrace{\frac{\partial \overline{u_1'^2 u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{2}{\rho} \frac{\partial \overline{p' u_1'}}{\partial x_1}}_{(3)} = \underbrace{\overline{\frac{2p'}{\rho} \frac{\partial u_1'}{\partial x_1}}}_{(4)}$$

$$- \underbrace{2\,\overline{u_1' u_k'} \frac{\partial \overline{u_1}}{\partial x_k}}_{(5)} + \underbrace{\nu \frac{\partial^2 \overline{u_1'^2}}{\partial x_k \partial x_k}}_{(6)} - \underbrace{2\nu \overline{\left( \frac{\partial u_1'}{\partial x_k} \right)^2}}_{(7)}$$

$$+ \underbrace{2f\,\overline{u_1' u_2'}}_{(9)}$$

# Transport equation for $\overline{v'^2}$

Take $i = j = 1$ in RS equation, $\overline{u_2' u_2'} = \overline{v'^2}$

## RS transport equation

$$\underbrace{\frac{\partial \overline{u_2'^2}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_2'^2}}{\partial x_k}}_{(1)} + \underbrace{\frac{\partial \overline{u_2'^2 u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{2}{\rho}\frac{\partial \overline{p' u_2'}}{\partial x_2}}_{(3)} = \underbrace{\overline{\frac{2p'}{\rho}\frac{\partial u_2'}{\partial x_2}}}_{(4)}$$

$$- \underbrace{2\,\overline{u_2' u_k'}\frac{\partial \overline{u_2}}{\partial x_k}}_{(5)} + \underbrace{\nu\frac{\partial^2 \overline{u_2'^2}}{\partial x_k \partial x_k}}_{(6)} - \underbrace{2\nu\overline{\left(\frac{\partial u_2'}{\partial x_k}\right)^2}}_{(7)}$$

$$- \underbrace{2f\,\overline{u_1' u_2'}}_{(9)}$$

# Transport equation for $\overline{w'^2}$

Take $i = j = 1$ in RS equation, $\overline{u_3' u_3'} = \overline{w'^2}$

## RS transport equation

$$\underbrace{\frac{\partial \overline{u_3'^2}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_3'^2}}{\partial x_k}}_{(1)} + \underbrace{\frac{\partial \overline{u_3'^2 u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{2}{\rho}\frac{\partial \overline{p' u_3'}}{\partial x_3}}_{(3)} = \underbrace{\overline{\frac{2p'}{\rho}\frac{\partial u_3'}{\partial x_3}}}_{(4)}$$

$$- \underbrace{2\,\overline{u_3' u_k'}\frac{\partial \overline{u_3}}{\partial x_k}}_{(5)} + \underbrace{\nu\frac{\partial^2 \overline{u_3'^2}}{\partial x_k \partial x_k}}_{(6)} - \underbrace{2\nu\overline{\left(\frac{\partial u_3'}{\partial x_k}\right)^2}}_{(7)}$$

$$+ \underbrace{2\,\overline{b' u_3'}}_{(8)}$$

# Transport equation for the turbulence kinetic energy

Take $i = j$ in RS equation, $k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\overline{u'^2 + v'^2 + w'^2}$

## RS transport equation

$$\underbrace{\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k}}_{(1)} + \frac{1}{2}\underbrace{\frac{\partial \overline{u_i' u_i' u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{1}{\rho}\frac{\partial \overline{p' u_i'}}{\partial x_i}}_{(3)} = \underbrace{\overline{\frac{p'}{\rho}\frac{\partial u_i'}{\partial x_i}}}_{=0}$$

$$- \underbrace{\overline{u_i' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}}_{(5)} + \nu\underbrace{\frac{\partial^2 k}{\partial x_k \partial x_k}}_{(6)} - \underbrace{\nu\overline{\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{(7)}$$

$$+ \underbrace{\overline{b' u_3'}}_{(8)} + \underbrace{f\overline{u_1' u_2'} - f\overline{u_1' u_2'}}_{=0}$$

# Transport equation for turbulence kinetic energy

Take $i = j$ in RS equation, $k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\overline{u'^2 + v'^2 + w'^2}$

## $k$ transport equation

$$\underbrace{\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k}}_{(1)} + \frac{1}{2}\underbrace{\frac{\partial \overline{u_i' u_i' u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{1}{\rho}\frac{\partial \overline{p' u_i'}}{\partial x_i}}_{(3)} =$$

$$- \underbrace{\overline{u_i' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}}_{(5)} + \underbrace{\nu\frac{\partial^2 k}{\partial x_k \partial x_k}}_{(6)} - \underbrace{\nu\overline{\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{(7)}$$

$$+ \underbrace{\overline{b' u_3'}}_{(8)}$$

# Dissipation rate of the turbulence kinetic energy

## Dissipation rate

$$\epsilon = -\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}}_{(7)}$$

## Transport equation for the dissipation rate

$$\frac{\partial \epsilon}{\partial t} + \overline{u_k}\frac{\partial \epsilon}{\partial x_k} = RHS \tag{1}$$

# Estimation of characteristic scales of turbulence

## Dimensional analysis

$$k = \frac{1}{2}\overline{u_i' u_i'} = \left[\frac{m^2}{s^2}\right], \quad \epsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)} = \left[\frac{m^2}{s^3}\right]$$

## Characteristic velocity, length and time scales

$$U \sim \sqrt{k}, \quad L \sim \frac{k^{3/2}}{\epsilon}, \quad \tau = \frac{L}{U}$$

$$L = C_\epsilon \frac{k^{3/2}}{\epsilon}$$

# Turbulence modelling

**The main idea of common RANS turbulence models is to replace the whole range of scales in turbulence by a single, characteristic scale $L$ with the corresponding velocity scale $U$.**

1. Turbulent (eddy) viscosity $\nu_t \sim UL$
2. Turbulent (eddy) diffusivity $\kappa_t = \nu_t / Pr_t$

$Pr_t = \mathcal{O}(1)$ is the turbulent Prandtl number

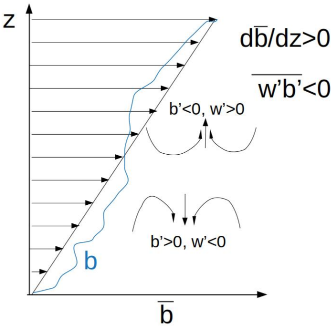
$\nu_t$, $\kappa_t$ and $Pr_t$ are the properties of the flow, not the material.

# Turbulence modelling

**Gradient diffusion hypothesis**

$$\overline{u_i' u_j'} - \frac{2}{3}\delta_{ij} k = -\nu_t \left( \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)$$

Assumes that the anisotropy tensor $a_{ij} = \overline{u_i' u_j'} - \frac{2}{3}\delta_{ij} k$ is aligned with the mean rate-of-strain tensor.




**Turbulent fluxes**

$$\overline{b' u_i'} = -\kappa_t \frac{\partial \overline{b}}{\partial x_i}$$

# Turbulence modelling

**0-equation models**

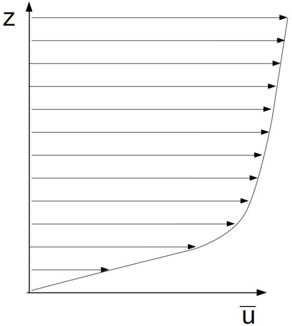
$$\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k = -\nu_t\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right)$$

with mixing length $l_m$

$$\nu_t = l_m^2\left|\frac{\partial\overline{u}}{\partial z}\right|$$

**Near the wall, in the so-called log-law region**

$$l_m \sim z$$



**1-equation models - Spallart-Allamaras model**

$$\overline{u_i' u_j'} - \frac{2}{3}\delta_{ij} k = -\nu_t \left( \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)$$

$+$ transport equation for $\nu_t$

$$\frac{\partial \nu_t}{\partial t} + \overline{u_k} \frac{\partial \nu_t}{\partial x_k} = RHS \qquad (2)$$

**Near the wall, in the so-called log-law region**

$$l_m \sim z$$

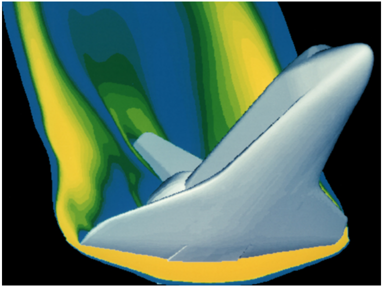
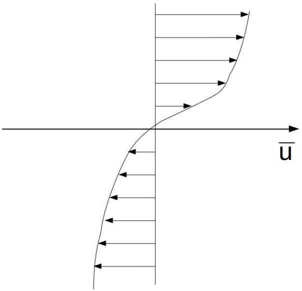


Figure: Source: https://www.nasa.gov/multimedia/imagegallery/image_feature_431.h (NASA - public domain)

**1-equation models - k-equation**

$$\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k = -\nu_t\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right)$$




with

$$\nu_t = ck^{1/2}l_m,$$

$+$ transport equation for $k$

$$\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k} = RHS$$

with dissipation

$$\epsilon = C_D k^{3/2}/l_m$$

# Turbulence modelling

## $k$ transport equation

$$\underbrace{\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k}}_{(1)} + \frac{1}{2}\underbrace{\frac{\partial \overline{u_i' u_i' u_k'}}{\partial x_k}}_{(2)} + \underbrace{\frac{1}{\rho}\frac{\partial \overline{p' u_i'}}{\partial x_i}}_{(3)} = -\underbrace{\overline{u_i' u_k'}\frac{\partial \overline{u_i}}{\partial x_k}}_{(5)} + \nu\underbrace{\frac{\partial^2 k}{\partial x_k \partial x_k}}_{(6)} - \epsilon$$

## Model for $k$ transport

$$\underbrace{\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k}}_{(1)} - \underbrace{\frac{\partial}{\partial x_k}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial x_k}\right)}_{(2)+(3)} = \underbrace{2\nu_t\frac{\partial \overline{u_i}}{\partial x_k}\frac{\partial \overline{u_i}}{\partial x_k}}_{(5)} + \nu\underbrace{\frac{\partial^2 k}{\partial x_k \partial x_k}}_{(6)} - \epsilon$$

# Turbulence modelling

**2-equation models - k-$\epsilon$**

$$\overline{u_i' u_j'} - \frac{2}{3}\delta_{ij} k = -\nu_t \left( \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)$$

with

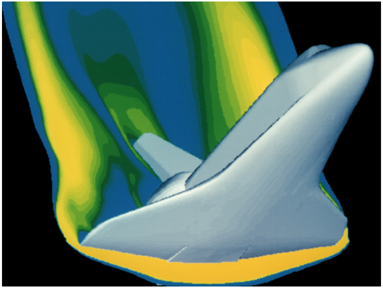
$$\nu_t = c_\mu \frac{k^2}{\epsilon},$$

+ transport equation for $k$

$$\frac{\partial k}{\partial t} + \overline{u_k} \frac{\partial k}{\partial x_k} = RHS$$

+ transport equation for $\epsilon$

$$\frac{\partial \epsilon}{\partial t} + \overline{u_k} \frac{\partial \epsilon}{\partial x_k} = RHS$$



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# Reynolds-stress modelling

**RS models, no eddy viscosity assumption**

$$\overline{u_i'u_j'}$$

$+$ transport equation for $\overline{u_i'u_j'}$

$$\frac{\partial \overline{u_i'u_j'}}{\partial t} + \overline{u_k}\frac{\partial \overline{u_i'u_j'}}{\partial x_k} = RHS$$

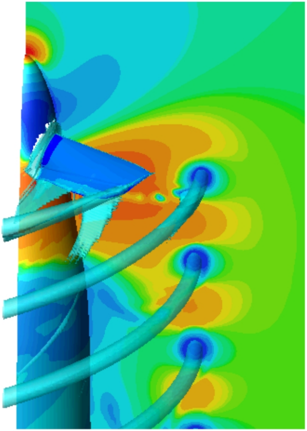


Figure: CFD simulation showing vorticity isosurfaces behind propeller. Author: User:Citizenthom, Source: Wikipedia, BB-CY-3.0

**algebraic RS models**

$$\frac{\overline{u'_i u'_j}}{k}(P - \epsilon) = RHS$$

$+$ transport equations for $k$ and $\epsilon$

$$\frac{\partial k}{\partial t} + \overline{u_k}\frac{\partial k}{\partial x_k} = RHS$$

$$\frac{\partial \epsilon}{\partial t} + \overline{u_k}\frac{\partial \epsilon}{\partial x_k} = RHS$$

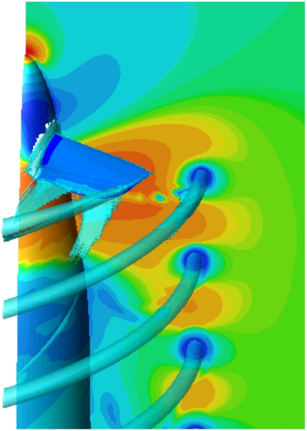


Figure: CFD simulation showing vorticity isosurfaces behind propeller. Author: User:Citizenthom, Source: Wikipedia, BB-CY-3.0

# Reynolds-stress modelling - simplifications

## Statistical stationarity

Averaged quantities, like e.g. mean velocity, Reynolds stresses etc. do not change with time

$$\frac{\partial \overline{Q}}{\partial t} = 0$$

## Statistical homogeneity

Averaged quantities, like e.g. mean velocity, Reynolds stresses etc. do not depend on position

$$\frac{\partial \overline{Q}}{\partial x} = 0, \quad \text{or/and} \quad \frac{\partial \overline{Q}}{\partial y} = 0 \quad \text{or/and} \quad \frac{\partial \overline{Q}}{\partial z} = 0$$

# Reynolds-stress modelling - simplifications

## Isotropy

Averaged quantities, like e.g. two-point correlations etc. do not depend on direction, but only on the module of the distance vector between points $|\boldsymbol{r}| = r$

$$R(\boldsymbol{x}, \boldsymbol{r}, t) = \overline{u(\boldsymbol{x}, t)u(\boldsymbol{x} + \boldsymbol{r}, t)} = R(r, t)$$

With these simplifications, analysis of the governing PDE becomes somewhat simpler.
Remember that the instantaneous quantities, like the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$, buoyancy $b(\boldsymbol{x}, t)$ etc. are fluctuating. The stationarity, homegeneity and isotropy concerns the ensemble averaged quantities!

# Reynolds-stress modelling - simplified equations

Consider a model for turbulence kinetic energy equation under the homogeneity assumption

## Homogeneity in $x_1 = x$, $x_2 = y$ and $x_3 = z$ directions

$$\frac{\partial k}{\partial t} + \underbrace{\overline{u_k}\frac{\partial k}{\partial x_k}}_{=0} - \underbrace{\frac{\partial}{\partial x_k}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial x_k}\right)}_{=0} = \underbrace{2\nu_t \frac{\partial \overline{u_i}}{\partial x_k}\frac{\partial \overline{u_i}}{\partial x_k}}_{=0} + \underbrace{\nu \frac{\partial^2 k}{\partial x_k \partial x_k}}_{=0} - \epsilon$$

## Homogeneity in $x_1 = x$, $x_2 = y$ and $x_3 = z$ directions

$$\frac{\partial k}{\partial t} = -\epsilon$$

This equation describes freely decaying turbulence. The dissipation $\epsilon$ still requires a model.

# Reynolds-stress modelling - simplified equations

## $k$-equation with homogeneity assumption

$$\frac{\partial k}{\partial t} = -\epsilon$$

This equation describes freely decaying turbulence. The dissipation $\epsilon$ still requires a model, so the solution is far from obvious even in this simplified case. E.g. Launder-Sharma model with simplifications due to homogeneity reads

## $\epsilon$ -equation with homogeneity assumption

$$\frac{\partial \epsilon}{\partial t} = -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

where $C_{\epsilon 2} = 1.92$.

# Reynolds-stress modelling - simplified equations

## $k - \epsilon$ -equation with homogeneity assumption

$$\frac{\partial k}{\partial t} = -\epsilon, \quad \frac{\partial \epsilon}{\partial t} = -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

In this case $k - \epsilon$ -equations have an analytical solution

## $k - \epsilon$ -equations solution

$$k(t) = k_0 \left(\frac{t}{t_0}\right)^{-n}, \quad \epsilon(t) = \epsilon_0 \left(\frac{t}{t_0}\right)^{-(n+1)}, \quad n = \frac{1}{C_{\epsilon 2} - 1}$$

Experimental observations suggest $n \approx 1.15 - 1.45$.

# RS modelling - free shear flows

Let us consider stationary, free-shear flows, that is, inhomogeneous flows far from boundaries, with a mean velocity gradient.
The simplifications for averaged quantities are

## free-shear flows - simplifications

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0, \quad \overline{v} = 0, \quad \overline{w} = 0$$

additionally, stationarity can be assumed

$$\frac{\partial}{\partial t} = 0$$

# RS modelling - free shear flows

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0, \quad \overline{v} = 0, \quad \overline{w} = 0$$

$$\frac{\partial}{\partial z} \neq 0,$$

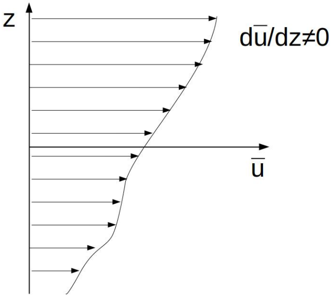



Figure: Free shear flow

# RS modelling - simplified equations, free shear flows

## $k$ - equation

$$\frac{\partial k}{\partial t} + \overline{u}\underbrace{\frac{\partial k}{\partial x}}_{=0} + \underbrace{\overline{v}}_{=0}\underbrace{\frac{\partial k}{\partial y}}_{=0} + \underbrace{\overline{w}}_{=0}\frac{\partial k}{\partial z}$$

$$-\underbrace{\frac{\partial}{\partial x}}_{=0}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial x}\right) - \underbrace{\frac{\partial}{\partial y}}_{=0}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial z}\right)$$

$$= \underbrace{2\nu_t\frac{\partial \overline{u_i}}{\partial x}\frac{\partial \overline{u_i}}{\partial x}}_{=0} + \underbrace{2\nu_t\frac{\partial \overline{u_i}}{\partial y}\frac{\partial \overline{u_i}}{\partial y}}_{=0} + 2\nu_t\frac{\partial \overline{u_i}}{\partial z}\frac{\partial \overline{u_i}}{\partial z}$$

$$+\nu\underbrace{\frac{\partial^2 k}{\partial x^2}}_{=0} + \nu\underbrace{\frac{\partial^2 k}{\partial y^2}}_{=0} + \nu\frac{\partial^2 k}{\partial z^2} - \epsilon$$

# RS modelling - simplified equations, free shear flows

**$k-\epsilon$ - equations**

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial z}\right) + \underbrace{2\nu_t \frac{\partial \overline{u_i}}{\partial z}\frac{\partial \overline{u_i}}{\partial z}}_{\mathcal{P}} + \nu\frac{\partial^2 k}{\partial z^2} - \epsilon$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial z}\right) + C_{\epsilon 1}\frac{\epsilon}{k}\mathcal{P} - C_{\epsilon 2}\frac{\epsilon^2}{k}$$

with $C_{\epsilon 1} = 1.44$, $C_{\epsilon 2} = 1.92$, $\sigma_\epsilon = 1.0$,

# RS modelling - simplified equations, free shear flows + homogeneity

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0, \quad \overline{v} = 0, \quad \overline{w} = 0$$

Only $\overline{u}$ depends on $z$ coordinate, such that

$$\frac{\partial \overline{u}}{\partial z} = const,$$

for remaining statistics

$$\frac{\partial}{\partial z} = 0.$$

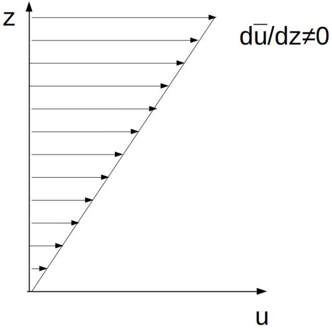



Figure: Free shear flow

# RS modelling - simplified equations, free shear flows + homogeneity

$$\frac{\partial}{\partial z} = 0, \quad \text{apart from} \quad \frac{\partial \overline{u}}{\partial z} = const., \quad \mathcal{P} = const,$$

## $k - \epsilon$ - equations

$$\begin{aligned} \frac{\partial k}{\partial t} &= \mathcal{P} - \epsilon \\ \frac{\partial \epsilon}{\partial t} &= C_{\epsilon 1} \frac{\epsilon}{k} \mathcal{P} - C_{\epsilon 2} \frac{\epsilon^2}{k} \end{aligned}$$

with $C_{\epsilon 1} = 1.44$, $C_{\epsilon 2} = 1.92$, $\sigma_{\epsilon} = 1.0$,

# RS modelling - simplified equations, free shear flows + homogeneity

Reynolds-stresses become self-similar, the non-dimensional parameter $\mathcal{P}/\epsilon$ becomes a constant

## Turbulence time scale $\tau = k/\epsilon$

$$\frac{\partial}{\partial t}\frac{k}{\epsilon} = (C_{\epsilon 2} - 1) - (C_{\epsilon 1} - 1)\frac{\mathcal{P}}{\epsilon}$$

Stationary case:

$$0 = (C_{\epsilon 2} - 1) - (C_{\epsilon 1} - 1)\frac{\mathcal{P}}{\epsilon}$$

Which results in

$$\frac{\mathcal{P}}{\epsilon} = \frac{(C_{\epsilon 2} - 1)}{(C_{\epsilon 1} - 1)} = \frac{0.92}{0.44} \approx 2.09$$

# RS modelling - simplified equations - shear flow + buoyancy

## $k - \epsilon$ - equations

$$
\begin{aligned}
\frac{\partial \overline{u}}{\partial t} &= -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} - \frac{\partial}{\partial z}\left(\nu_t \frac{\partial \overline{u}}{\partial z}\right) + \nu \frac{\partial^2 \overline{u}}{\partial z^2} \\
0 &= -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial z} + \overline{b} \\
\frac{\partial \overline{b}}{\partial t} &= -\frac{\partial}{\partial z}\left(\frac{\nu_t}{Pr_t}\frac{\partial \overline{b}}{\partial z}\right) + \nu \frac{\partial^2 \overline{b}}{\partial z^2} \\
\frac{\partial k}{\partial t} &= \frac{\partial}{\partial z}\left(\frac{\nu_t}{Pr_t}\frac{\partial k}{\partial z}\right) + \mathcal{P} + \nu \frac{\partial^2 k}{\partial z^2} - \epsilon - \frac{\nu_t}{Pr_t}\frac{\partial \overline{b}}{\partial z} \\
\frac{\partial \epsilon}{\partial t} &= \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial z}\right) + C_{\epsilon 1}\frac{\epsilon}{k}\mathcal{P} - C_{\epsilon 2}\frac{\epsilon^2}{k}
\end{aligned}
$$

# Large eddy simulations

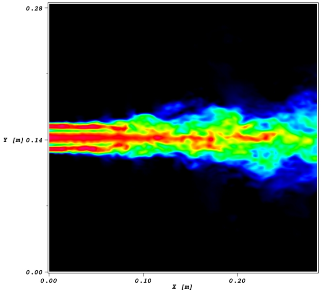



Figure: Large Eddy Simulation of turbulent jet. Source:
https://commons.wikimedia.org/wiki/File:LES_Turbulent_Velocity_Field.png,
Author: Charlesreid1, CC-BY-SA 3.0

# Large eddy simulations - filtering

Filtering operation

$$\widetilde{Q}(\boldsymbol{x}, t) = \int Q(\boldsymbol{x} - \boldsymbol{r}, t) G(\boldsymbol{r}, \boldsymbol{x}) d\boldsymbol{r}$$

where

$$G(\boldsymbol{x}', \boldsymbol{x}), \quad \text{is a filter function, such that}$$

$$\int G(\boldsymbol{r}, \boldsymbol{x}) \mathrm{d}\boldsymbol{r} = 1.$$

Homogeneous filter (does not depend on position) $G = G(\boldsymbol{r})$
Isotropic filter (does not depend on position) $G = G(r)$

# Large eddy simulations - filtering

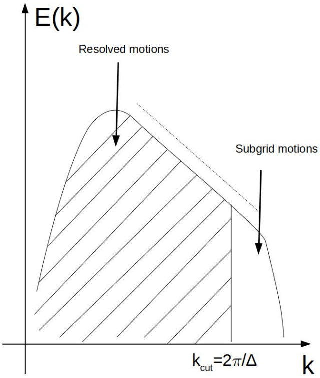



Figure: Resolved and subgrid part of the energy spectrum

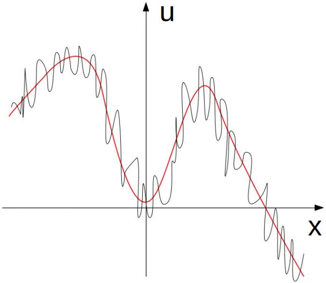



Figure: Instantaneous (black line) and filtered velocity (red line) .

# Box filter

$$G(r) = \frac{1}{\Delta} H(\frac{1}{2}\Delta - |\boldsymbol{r}|)$$

where $H$ is a Heaviside function.

In the Fourier space

$$\hat{G}(\kappa) = \int \mathrm{e}^{i\kappa r} G(r) dr = \frac{sin(\frac{1}{2}\kappa\Delta)}{\frac{1}{2}\kappa\Delta}$$

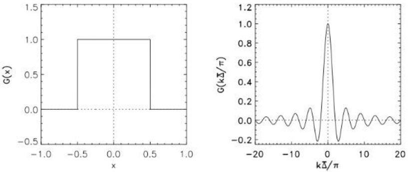



Figure: Box filter https://en.wikipedia.org/wiki/Filter_(large_eddy_simulation), Author: Charlesreid1, CC-BY-SA-4.0.

# Gaussian filter

$$G(r) = \left(\frac{6}{\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{6r^2}{\Delta^2}\right)$$

where $H$ is a Heaviside function.

In the Fourier space

$$\hat{G}(\kappa) = \int \mathrm{e}^{i\kappa r} G(r) dr = \exp\left(-\frac{\kappa^2\Delta^2}{24}\right)$$

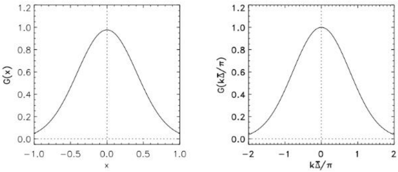



Figure: Gaussian filter https://en.wikipedia.org/wiki/Filter_(large_eddy_simulation), Author: Charlesreid1, CC-BY-SA-4.0.

$$G(r) = \frac{sin(\pi r/\Delta)}{\pi r}$$

where $H$ is a Heaviside function.
In the Fourier space

$$\hat{G}(\kappa) = \int \mathrm{e}^{i\kappa r} G(r) dr = H(\frac{\pi}{\Delta} - |\kappa|)$$

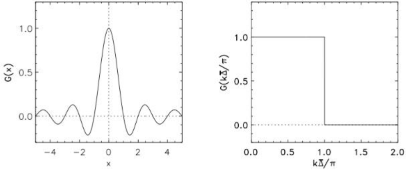



Figure: Sharp spectral filter https://en.wikipedia.org/wiki/Filter_(large_eddy_simulation), Author: Charlesreid1, CC-BY-SA-4.0.

# Large eddy simulations - filtering

Decomposition of velocity filed

$$\boldsymbol{u}(\boldsymbol{x},t) = \underbrace{\widetilde{\boldsymbol{u}(\boldsymbol{x},t)}}_{filtered\ velocity} + \underbrace{\boldsymbol{u}'(\boldsymbol{x},t)}_{subgrid\ (residual)\ velocity}$$

Properties of the filtering operator

$$\widetilde{Q+P} = \widetilde{Q} + \widetilde{P}, \quad \widetilde{\frac{\partial Q}{\partial t}} = \frac{\partial \widetilde{Q}}{\partial t}$$

If the filter is homogeneous, then

$$\widetilde{\frac{\partial Q}{\partial x_i}} = \frac{\partial \widetilde{Q}}{\partial x_i}$$

But

$$\widetilde{\widetilde{Q}} \neq \widetilde{Q}, \quad \text{hence} \quad \widetilde{Q'} = \widetilde{Q - \widetilde{Q}} \neq 0$$

# Filtered equations

## Momentum

$$\frac{\partial \widetilde{u}_i}{\partial t} + \widetilde{u}_j \frac{\partial \widetilde{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \underbrace{\left(\widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j\right)}_{\tau_{ij}^r} = -\frac{1}{\rho} \frac{\partial \widetilde{p}}{\partial x_i} + \nu \frac{\partial^2 \widetilde{u}_i}{\partial x_j \partial x_j} + \delta_{i3} \widetilde{b} + \epsilon_{ij3} f \widetilde{u}_j$$

## Buoyancy

$$\frac{\partial \widetilde{b}}{\partial t} + \widetilde{u}_j \frac{\partial \widetilde{b}}{\partial x_j} + \frac{\partial}{\partial x_j} \underbrace{\left(\widetilde{b u_j} - \widetilde{b} \widetilde{u}_j\right)}_{q_i^r} = \kappa \frac{\partial^2 \widetilde{b}}{\partial x_j \partial x_j}$$

## Continuity

$$\frac{\partial \widetilde{u}_i}{\partial x_i} = 0$$

# Smagorinsky model

## Residual stresses

$$\tau_{ij}^r = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j$$

Deviatoric part of $\tau_{ij}^r$

$$\tau_{ij}^d = \tau_{ij}^r - \frac{1}{3}\tau_{ii}^r \delta_{ij} = -2\nu_r \widetilde{S_{ij}} = -\nu_r \left( \frac{\partial \widetilde{u}_i}{\partial x_j} + \frac{\partial \widetilde{u}_j}{\partial x_i} \right)$$

$$\nu_r = l_s^2 \left( \widetilde{S_{ij}} \widetilde{S_{ij}} \right)^{1/2} = (C_s \Delta)^2 \mathcal{S}$$

So, it is assumed that the Smagorinsky length scale is proportional to the filter width $l_s = C_s \Delta$

# Decomposition of residual stresses

Proposed by Germano (1986)

## Residual stresses

$$\tau^r_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j, \quad u_i = \widetilde{u}_i + u'_i$$

$$\tau^r_{ij} = \mathcal{L}_{ij} + \mathcal{C}_{ij} + \mathcal{R}_{ij}$$

# Decomposition of residual stresses

Proposed by Germano (1986)

## Leonard stresses

$$\mathcal{L}_{ij} = \widetilde{\tilde{u}_i \tilde{u}_j} - \tilde{\tilde{u}}_i \tilde{\tilde{u}}_j,$$

## cross stresses

$$\mathcal{C}_{ij} = \widetilde{\tilde{u}_i u'_j} + \widetilde{\tilde{u}_j u'_i} - \tilde{\tilde{u}}_i \tilde{u}'_j - \tilde{\tilde{u}}_j \tilde{u}'_i$$

## SGS Reynolds stresses

$$\mathcal{R}_{ij} = \widetilde{u'_i u'_j} - \tilde{u}'_i \tilde{u}'_j$$

# Germano model

Let us consider two filters, one of width $\Delta$ and a second one of width $2\Delta$
Test filtering operation will be denoted by $\hat{\cdot}$.

## Filter and test filter

$$\tau^r_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j,$$

$$T^r_{ij} = \widehat{\widetilde{u_i u_j}} - \widehat{\widetilde{u}}_i \widehat{\widetilde{u}}_j,$$

Let us apply the test filtering operation to $\tau^r_{ij}$ and subtract $\widehat{\tau^r_{ij}}$ from $T^r_{ij}$

## Leonard stresses

$$\mathcal{L}_{ij} = T^r_{ij} - \widehat{\tau^r_{ij}} = \widehat{\widetilde{u}_i \widetilde{u}_j} - \widehat{\widetilde{u}}_i \widehat{\widetilde{u}}_j$$

# Germano model

Let us consider two filters, one of width $\Delta$ and a second one of width $\hat{\Delta} = 2\Delta$ Test filtering operation will be denoted by $\hat{Q}$.

## Model

$$\tau^a_{ij} = -2c_s\Delta^2\widetilde{\mathcal{S}}\,\widetilde{S_{ij}}$$

$$T^a_{ij} = -2c_s\hat{\Delta}^2\widehat{\widetilde{\mathcal{S}}}\,\widehat{\widetilde{S_{ij}}},$$

Let us apply the test filtering operation to $\tau^r_{ij}$ and subtract $\widehat{\tau^r_{ij}}$ from $T^r_{ij}$

## Leonard stresses

$$\mathcal{L}^a_{ij} = T^a_{ij} - \widehat{\tau^a_{ij}} = 2c_s\Delta^2\widehat{\widetilde{\mathcal{S}}\,\widetilde{S_{ij}}} - 2c_s\hat{\Delta}^2\widehat{\widetilde{\mathcal{S}}}\,\widehat{\widetilde{S_{ij}}} = c_s M_{ij}$$

In LES both $\mathcal{L}^a_{ij}$ and $M_{ij}$ are known in terms of $\widetilde{u}$, so this relation can be used to determine local value of $c_s$.

# Germano model

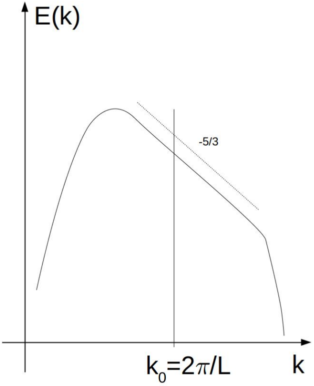
Determine $c_s$ by the local minimalization procedure

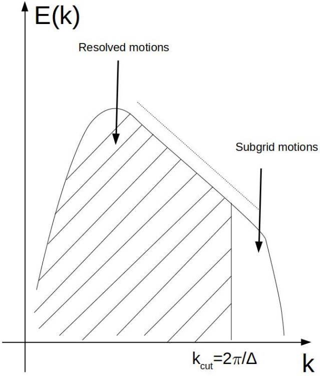
**Model**

$$c_S(\boldsymbol{x}, t) = M_{ij}\mathcal{L}_{ij}/M_{kl}M_{kl}$$

Values of $c_S < 0$ are possible (although computationally unstable) - indication of backscatter. In practice $c_S$ is set to $c_S = 0$ in such situation.

# LES vs. RANS




RANS - one length scale $L$ and one velocity scale $U \sim \sqrt{k}$, $k = \int E(\kappa) d\kappa$

LES - resolved scales $\kappa < \kappa_{cut}$ and subgrid scales for wavenumbers $\kappa < \kappa_{cut}$

# References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An introduction to turbulence
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# The End