# Turbulence and atmospheric boundary layer
# Lecture 7

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# Summary of lecture 6

1. Balance of energy in the spectral space
2. Balance of energy with forcing function
3. Structure of small scale turbulence
   - Internal intermittency
   - PDF's of velocity derivative
   - PDF's of velocity difference
   - Anomalous scaling
4. External intermittency

# 2D turbulence

Figure: Vortices in the Jupiter atmosphere. Juno mission. From https://www.missionjuno.swri.edu/

# 2D turbulence

Figure: Soap film. Sourec: https://en.wikipedia.org/wiki/Soap_bubble, Author: Brocken Inaglory, CC BY-Sa 3.0

Figure: From: https://www.labroots.com/tag/atmospheric-turbulence

Flows in the atmosphere - quasi 2D: horizontal length scale much larger than the vertical one.
3D/2D problem?

# Vorticity formulation

Let us consider the Navier-Stokes equation in the vorticity formulation

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad \omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

For this, calculate the *curl* of the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}.$$

We obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega}$$

Note that $\nabla \times \nabla p = 0$. What about

$$\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u})?$$

# Vorticity formulation

Check that

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla\left(\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times\boldsymbol{\omega}$$

Then

$$\begin{aligned}\nabla\times\left(\mathbf{u}\cdot\nabla\mathbf{u}\right) &= \underbrace{\nabla\times\frac{1}{2}\nabla\left(\mathbf{u}\cdot\mathbf{u}\right)}_{=0} - \nabla\times\left(\mathbf{u}\times\boldsymbol{\omega}\right)\\ &= \left(\mathbf{u}\cdot\nabla\right)\boldsymbol{\omega} - \left(\boldsymbol{\omega}\cdot\nabla\right)\mathbf{u} + \underbrace{\boldsymbol{\omega}\left(\nabla\cdot\mathbf{u}\right)}_{=0} + \underbrace{\mathbf{u}\left(\nabla\cdot\boldsymbol{\omega}\right)}_{=0}.\end{aligned}$$

Note that

$$\boldsymbol{\omega}\left(\nabla\cdot\mathbf{u}\right) = 0, \quad \text{due to continuity,}$$

$$\mathbf{u}\left(\nabla\cdot\boldsymbol{\omega}\right) = 0, \quad \text{because divergence of a curl is zero,}$$

# Vorticity formulation

Finally, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\boldsymbol{\omega} = \underbrace{(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u}}_{\text{vortex stretching term}} + \nu \nabla^2 \boldsymbol{\omega}$$

The vortex stretching term $(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u}$ is responsible for the increase of the vorticity magnitude due to velocity gradients. This is a very important process in turbulent flows.
Let us check what happens with this term in 2D turbulence?

$$\mathbf{u} = [u, v, 0], \quad \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right]$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = [0, 0, \omega]$$

Hence,

$$(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} = 0!!!$$

# Vorticity formulation

Hence, in 2D turbulence

$$(\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} = 0!!!$$

The mechanism of the vortex stretching is absent in 2D turbulence! There is an idea (although debated) that this mechanism is responsible for the downward energy cascade in 3D turbulence. The absence of the vortex stretching in 2D turbulence shows the physics of the two cases must also differ substantially.

**Vortex stretching**

https://www.youtube.com/watch?v=JaABLY6E8HE

# Energy cascade in 2D turbulence

In the 2D turbulence we have

**Transport equation for vorticity in 2D turbulence**

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}$$

In the case of zero viscosity

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} = 0$$

This, from the Lagrangian point of view means that the fluid element conserves its vorticity.

$$\frac{d\omega}{dt} = 0.$$

# Energy cascade in 2D turbulence

Let us consider homogeneous, isotropic turbulence without mean gradients. The kinetic energy reads

$$k = \frac{1}{2}\overline{\mathbf{u}^2} = \frac{1}{2}\int \mathbf{u}^2 dV = \int E(\kappa, t)d\kappa$$

But, as the vorticity is also conserved in the inviscid case, let us define another quantity, the enstrophy

$$\mathcal{E} = \frac{1}{2}\overline{\omega^2} = \frac{1}{2}\int \omega^2 dV = \int \kappa^2 E(\kappa, t)d\kappa$$

We have

$$\frac{dk}{dt} = -\nu\mathcal{E}$$

end

$$\frac{d\mathcal{E}}{dt} = -\nu\overline{(|\nabla\omega|)^2}$$

# Energy cascade in 2D turbulence

## Energy and enstrophy

$$\frac{dk}{dt} = -\nu\mathcal{E}, \quad \frac{d\mathcal{E}}{dt} = -\nu\overline{(|\nabla\omega|)^2}$$

In the inviscid case $\nu = 0$

## Energy and enstrophy at $\nu = 0$

$$\frac{dk}{dt} = 0, \quad \frac{d\mathcal{E}}{dt} = 0$$

We have two quadratic inviscid invariants.

This has very important consequences.

# 3D vs. 2D turbulence

## 3D turbulence

$$\frac{dk}{dt} = -\nu\mathcal{E},$$

versus

## 2D turbulence

$$\frac{dk}{dt} = -\nu\mathcal{E}, \quad \frac{d\mathcal{E}}{dt} = -\nu\overline{(|\nabla\omega|)^2}$$

What happens at $\nu \to 0$?

# 3D vs. 2D turbulence: What happens at $\nu \to 0$?

## 3D turbulence - dissipation anomaly

$$\frac{dk}{dt} = -\nu\mathcal{E},$$

Recall from the previous lecture that when $\nu \to 0$, then $\mathcal{E} \to \infty$.

versus

## 2D turbulence - no dissipation anomaly!

$$\frac{dk}{dt} = -\nu\mathcal{E}, \quad \frac{d\mathcal{E}}{dt} = -\nu\overline{(|\nabla\omega|)^2}$$

When $\nu \to 0$ then $\nu\overline{(|\nabla\omega|)^2} \to const$ (enstrophy dissipation anomaly), hence $d\mathcal{E}/dt < 0$ and eventually

$$\nu\mathcal{E} \to 0, \quad \frac{dk}{dt} \to 0.$$

# 3D vs. 2D turbulence

No dissipation anomaly means the energy cannot be dissipated at small scales.

In the numerical simulations, to obtain stationary spectra the following can be used

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} - \alpha \omega + f$$

where $f$ is a forcing function applied at small scales of order $l_f$

and $-\alpha\omega$ is a large scale friction term that extracts energy at large scales of order $\eta_{fr}$.

Inertial range is observed at scales

$$l_f << r << \eta_{fr}$$

# 3D vs. 2D turbulence.

As there is no dissipation anomaly, we expect there is no anomalous scaling in 2D turbulence.

Recall that in 3D turbulence, deviations from the Kolmogorov's scaling are observed. They are more pronounced for large $n$. Anomalous scaling in 3D turbulence

$$S^{(n)} \sim (\epsilon r)^{\zeta(n)}$$

where

$$\zeta(n) \neq n/3$$

for large $n$.

Figure: [J. Jiménez, Encyclopedia of Mathematical Physics, 144-151, 2006]

Figure: G. Boffetta, A. Celani and M. Vergassola, Inverse energy cascade in two-dimensional turbulence:Deviations from Gaussian behavior Physical Review E61, R29 (2000)

As there is no dissipation anomaly, we expect there is no anomalous scaling in 2D turbulence for the velocity structure functions.

In 2D turbulence the velocity structure functions follow the Kolmogorov's scaling in the inertial range

$$S^{(n)} \sim (\epsilon r)^{n/3}$$

# 3D vs. 2D turbulence. PDF's of velocity differences

Figure: 3D turbulence - non-Gaussian tails of the PDF's due to internal intermittency. From [Gotoh et al., Physics of Fluids 14(3), 2002]

Figure: 2D turbulence - PDF's close to Gaussian - no internal intermittency [G. Boffetta, A. Celani and M. Vergassola, Physical Review E61, R29 (2000)]

# Double cascade in 2D turbulence.

In the simulations of [G. Boffetta, A. Celani and M. Vergassola, Physical Review E61, R29 (2000)] forcing was applied at small scales.
What if the forcing is applied at intermediate scales $l_F = 2\pi/\kappa_F$?

## Kreichnan's double energy scenario

K. H. Kreichnan *Inertial Ranges in Two Dimensional Turbulence. Physics of Fluids 10, 1417 (1967)*

# Double cascade in 2D turbulence.

Recall that in 3D turbulence, exact result exists for the third order structure function:

$$S_{LLL} = \overline{(u_L(\boldsymbol{x}, t) - u_L(\boldsymbol{x} + \boldsymbol{r}, t))^3}$$

From the Kármán-Howarth equation at $\nu \to 0$ and in the steady case we obtain

**Kolmogorov's 4/5 law in 3D turbulence**

$$S_{LLL} = -\frac{4}{5}\epsilon r$$

Analogous relation can be derived for 2D turbulence for the velocity structure function

**Kolmogorov's 3/2 law in 3D turbulence**

$$S_{LLL} = \frac{3}{2}\epsilon r$$

# Double cascade in 2D turbulence.

**Valid for $\kappa << \kappa_F$**

$$S_{LLL} = \frac{3}{2}\epsilon r$$

As the enstrophy is another quadratic invariant in 2D turbulence, hence analogous relation can be obtained for the enstrophy structure functions in 2D turbulence

$$S_{L\omega\omega} = \overline{(u_L(\boldsymbol{x}, t) - u_L(\boldsymbol{x} + \boldsymbol{r}, t))\,(\omega(\boldsymbol{x}, t) - \omega(\boldsymbol{x} + \boldsymbol{r}, t))^2}$$

**Valid for $\kappa >> \kappa_F$**

$$S_{L\omega\omega} = -\frac{4}{3}\zeta r$$

where $\zeta = \nu\overline{(|\nabla\omega|)^2}$ is the palinstrophy.

# Double cascade in 2D turbulence.

The sign of the third-order structure function is related to the direction of the energy transfer. This can be deduced from the Fourier transform of the Kármán-Howarth equation. It follows that in 3D turbulence

$$S_{LLL} = -\frac{4}{5}\epsilon r < 0$$

indicates the transport of energy towards small scales. In 2D turbulence

$$S_{LLL} = \frac{3}{2}\epsilon r > 0$$

indicates the energy is transported from $\kappa_F$ towards larger scales (inverse energy cascade for $\kappa << \kappa_F$)

$$S_{L\omega\omega} = -\frac{4}{3}\zeta r < 0$$

indicates the enstrophy is transported from $\kappa_F$ towards smaller scales (direct enstrophy cascade for $\kappa >> \kappa_F$).

# Double cascade in 2D turbulence - dimensional analysis

At $\kappa << \kappa_F$ in the inertial range we have the inverse energy cascade with the constant flux $\epsilon$.

$$[\epsilon] = m^2/s^3$$

and

$$[E(\kappa)d\kappa] = m^2/s^2, \quad [E(\kappa)] = m^2/s^2$$

hence

$$E(\kappa)d\kappa \sim \epsilon^{2/3}\kappa^{-5/3}$$

At $\kappa >> \kappa_F$ in the inertial range we have the forward enstrophy cascade with the constant flux $\zeta$

$$[\zeta] = 1/s^3$$

hence

$$E(\kappa)d\kappa \sim \zeta^{2/3}\kappa^{-3}$$

# Double cascade in 2D turbulence - dimensional analysis

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hence

$$E(\kappa) \sim \zeta^{2/3}\kappa^{-3}$$

# Double cascade in 2D turbulence

So, we expect a scale break in the energy spectrum and the existence of two different inertial ranges

**Inverse energy cascade $L << \kappa << \kappa_F$**

$$E(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$$

**Forward enstrophy cascade cascade $\kappa_F << \kappa << \eta$**

$$E(\kappa) \sim \zeta^{2/3}\kappa^{-3}\ln^{-1/3}(\kappa/\kappa_F)$$

[Kraichnan, R. (1971). Inertial-range transfer in two- and three-dimensional turbulence. Journal of Fluid Mechanics, 47(3), 525-535]

# Double cascade in 2D turbulence

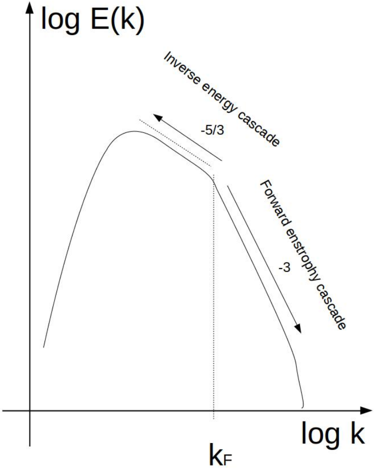


Figure: Double cascade

# Scale break in the atmospheric turbulence

Figure: From: G D Nastrom and K S Gage, A climatology of atmospheric wavenumber spectra of wind and temperature observed by commercial aircraft, Physics, 1985

# Scale break in the atmospheric turbulence

As follows from the aircraft measurements, [G D Nastrom and K S Gage, Physics, 1985] Kolmogorov spectrum is observed in the range

$$\lambda = 3 - 300km, \quad E(\kappa) \sim \kappa^{-5/3}$$

but there is an evidence of $\kappa^{-3}$ scaling at larger scales

$$\lambda > 400km, \quad E(\kappa) \sim \kappa^{-3}$$

Note that this is not the double cascade spectrum of 2D turbulence. There is an evidence of $\kappa^{-3}$ scaling at large scales that follows from the geostrophy and the forward energy cascade at scales $\lambda < 300km$ with direct energy cascade and the scaling $\kappa^{-5/3}$.
So it is a 2D/3D case.

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# The End