Turbulence and atmospheric boundary layer
Lecture 5

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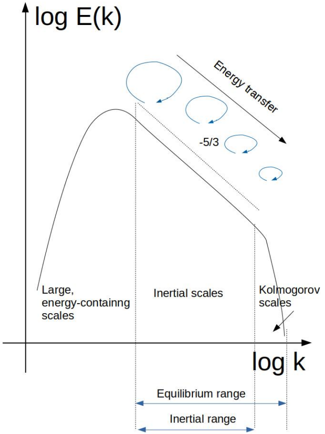
Summary of lecture 4

1. Transport equation for the turbulence kinetic energy
2. Turbulence modelling
   - 0-equation, algebraic models (mixing length), 1-equation models (e.g. Spallart-Allamaras), 2-equation models ($k - \epsilon$, $k - \omega$), Reynolds-stress models
3. Simplifications - statistical stationarity, statistical homogeneity, isotropy
4. Decaying turbulence
5. Free-shear flows
6. Large Eddy Simulations
   - filtering operation, properties of the filter, Smagorinsky model, Germano dynamic model

Structure of turbulence

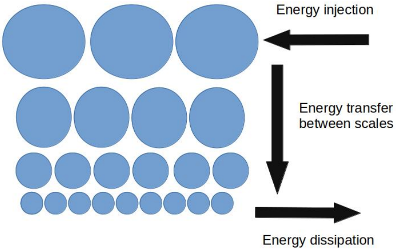
Energy cascade:
"Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity. "
Lewis Fry Richardson
(1881-1953)



Structure of turbulence

Energy cascade:
Dissipation is placed at the end of a sequence of processes.
Hence, the rate of dissipation ϵ is determined by the first process in the sequence, that is - transfer of energy from the largest eddies.

$$\epsilon \sim \frac{U^3}{L}$$



Kolmogorov's theory

L_0 - length scale of large, energy-containing eddies,
η - length scale of the smallest, dissipative eddies

Kolmogorov's hypothesis of local isotropy

At sufficiently large Reynolds number, the small scale turbulent motions are statistically isotropic

First Kolmogorov's similarity hypothesis

In every turbulent flow at sufficiently high Re the statistics of small-scale motions have a universal form that is uniquely determined by ν and ϵ.

Second Kolmogorov's similarity hypothesis

Second similarity hypothesis - in every turbulent flow at sufficiently high Re the statistics of the motions of scale l such that $L_0 >> l >> \eta$ have a universal form that is uniquely determined by ϵ, independent of ν.

Kolmogorov's theory

First Kolmogorov's similarity hypothesis

In every turbulent flow at sufficiently high Re the statistics of small-scale motions have a universal form that is uniquely determined by ν and ϵ.

Kolmogorov's scales

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \quad u_\eta = (\nu\epsilon)^{1/4}, \quad \tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}$$

Kolmogorov's scales, constructed based on the viscosity ν and the dissipation rate ϵ characterise the smallest dissipative eddies.

Structure functions

$$S_{ij}(\boldsymbol{x},\boldsymbol{r},t) = \overline{[u_i(\boldsymbol{x},t) - u_i(\boldsymbol{x}+\boldsymbol{r},t)]\,[u_j(\boldsymbol{x},t) - u_j(\boldsymbol{x}+\boldsymbol{r},t)]}$$

Let us identify the longitudinal velocity component along the vector $\boldsymbol{r}$ between the points: u_L and two transverse components u_{N1} and u_{N2}

longitudinal structure function

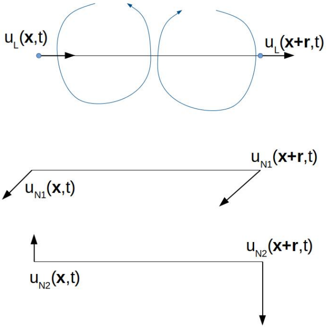
$$S_{LL} = \overline{[u_L(\boldsymbol{x},t) - u_L(\boldsymbol{x}+\boldsymbol{r},t)]^2}$$

transverse structure function

$$S_{N1N1} = \overline{[u_{N1}(\boldsymbol{x},t) - u_{N1}(\boldsymbol{x}+\boldsymbol{r},t)]^2}$$

transverse structure function

$$S_{N2N2} = \overline{[u_{N2}(\boldsymbol{x},t) - u_{N2}(\boldsymbol{x}+\boldsymbol{r},t)]^2}$$



Structure functions in isotropic turbulence

$$S_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = S_{ij}(r, t)$$

Moreover, $S_{N1N1} = S_{N2N2} = S_{NN}$ and $S_{ij} = 0$ for $i \neq j$.
Tensor calculus provides the following form of S_{ij}:

$$S_{ij}(\boldsymbol{r}, t) = S_{NN}\delta_{ij} + (S_{LL} - S_{NN})\frac{r_i r_j}{r^2} \qquad (1)$$

From the continuity equation it follows that

$$\frac{\partial S_{ij}}{\partial r_i} = 0$$

If we differentiate Eq. (1) with respect to $\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r}$ we obtain

$$S_{NN} = S_{LL} + \frac{1}{2} r \frac{\partial S_{LL}}{\partial r}$$

Hence, in the isotropic turbulence the structure function tensor is determined by a single scalar function S_{LL}.

Structure functions and Kolmogorov's theory

First Kolmogorov's similarity hypothesis

In every turbulent flow at sufficiently high Re the statistics of small-scale motions have a universal form that is uniquely determined by ν and ϵ.

$$S_{LL} = F(r, \epsilon, \nu)$$

Dimensional analysis: S_{ij} has the dimension of m^2/s^2, same as $(\epsilon r)^{2/3}$ so, let us create a non-dimensional function

$$\frac{S_{LL}}{(\epsilon r)^{2/3}} = F^{+}(r, \epsilon, \nu)$$

The only non-dimensional combination of r, ϵ, ν is $r\epsilon^{1/4}/\nu^{3/4} = r/\eta$
Hence,

$$S_{LL} = (\epsilon r)^{2/3} F^{+}\left(\frac{r}{\eta}\right)$$

Structure functions and Kolmogorov's theory

Second Kolmogorov's similarity hypothesis

Second similarity hypothesis - in every turbulent flow at sufficiently high Re the statistics of the motions of scale l such that $L_0 >> l >> \eta$ have a universal form that is uniquely determined by ϵ, independent of ν.

$$S_{LL} = (\epsilon r)^{2/3} F^{+} \left(\frac{r}{\eta} \right)$$

In the inertial subrange the function F^{+} is a constant and

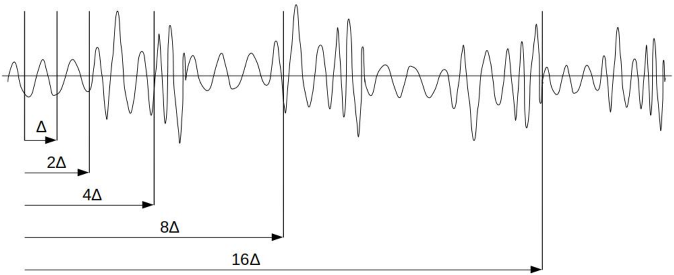
$$S_{LL} = C_2 (\epsilon r)^{2/3}, \quad S_{NN} = \frac{4}{3} C_2 (\epsilon r)^{2/3},$$

where $C_2 \approx 2$, as follows from measurements.

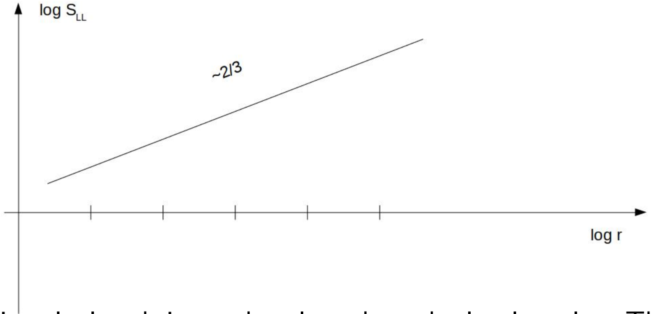
Structure functions and Kolmogorov's theory

$$S_{ij}(\boldsymbol{r}, t) = C_2(\epsilon r)^{2/3}\left(\frac{4}{3}\delta_{ij} - \frac{1}{3}\frac{r_i r_j}{r^2}\right),$$

Thus, in the isotropic turbulence, in the inertial range the Kolmogorov's hypotheses are sufficient to determine the dissipation rate ϵ (characteristic of small-scales) in terms of the second-order structure function S_{ij}, the distance r and the constant C_2.



Structure functions and Kolmogorov's theory



Once S_{LL} is calculated, it can be plotted on the log-log plot. The power-law function is a straight line on such plot with the power term $2/3$ corresponding to the slope, and the constant term $C_2\epsilon^{2/3}$ corresponding to the intercept of the line.

$$\log S_{LL} = \log r^{2/3} + \log C_2\epsilon^{2/3} = 2/3 \log r + \log C_2\epsilon^{2/3}$$

Structure functions - the Kármán-Howarth equation

From the Navier-Stokes equations, transport equations for S_{LL} can be derived. This is the so-called Kármán-Howarth equation. In the stationary case this equation reads

$$0 = 6\nu \frac{\partial S_{LL}}{\partial r} - S_{LLL} - \frac{4}{5}\epsilon r$$

where $S_{LLL} = \overline{(u_L(\boldsymbol{x}, t) - u_L(\boldsymbol{x} + \boldsymbol{r}, t))^3}$.
The viscous term is negligible in the inertial subrange, which leads to the

Kolmogorov's 4/5 law

$$S_{LLL} = \frac{4}{5}\epsilon r$$

This is an exact result (no experimental constant). Allows for a more precise estimation of ϵ in the isotropic case. (However, the size of the ensemble necessary to calculate S_{LLL} with a good accuracy is larger than for S_{LL}.)

Two-point correlations

$$R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = \overline{u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x} + \boldsymbol{r}, t)}$$

In the isotropic turbulence

$$R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = R_{ij}(r, t)$$

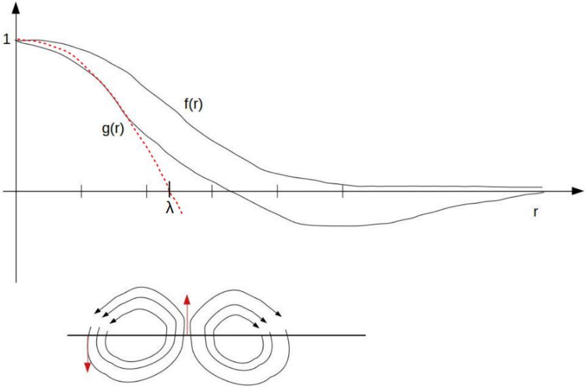
Longitudinal and transverse autocorrelation functions

$$f(r, t) = \frac{\overline{u_L(\boldsymbol{x}, t) u_L(\boldsymbol{x} + \boldsymbol{r}, t)}}{\overline{u_L^2(\boldsymbol{x}, t)}} = \frac{R_{LL}(r, t)}{\overline{u_L^2}}$$

$$g(r, t) = \frac{\overline{u_N(\boldsymbol{x}, t) u_N(\boldsymbol{x} + \boldsymbol{r}, t)}}{\overline{u_N^2(\boldsymbol{x}, t)}} = \frac{R_{NN}(r, t)}{\overline{u_N^2}}$$

where $\overline{u_L^2} = \overline{u_N^2} = u'^2$

Autocorrelation functions



Autocorrelation functions - length scales

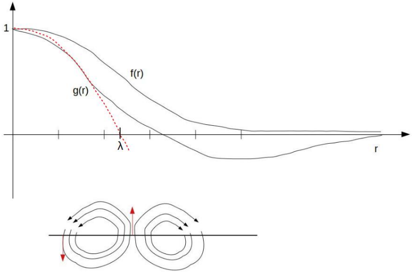
Integral length scale:

$$\mathcal{L} = \int_0^\infty f(r)dr$$

Taylor microscales

$$\lambda_f = \left[-\frac{1}{2} \frac{d^2 f(r)}{dr^2} \bigg|_{r=0} \right]^{-1/2}$$

$$\lambda_g = \left[-\frac{1}{2} \frac{d^2 g(r)}{dr^2} \bigg|_{r=0} \right]^{-1/2}$$



Energy spectra

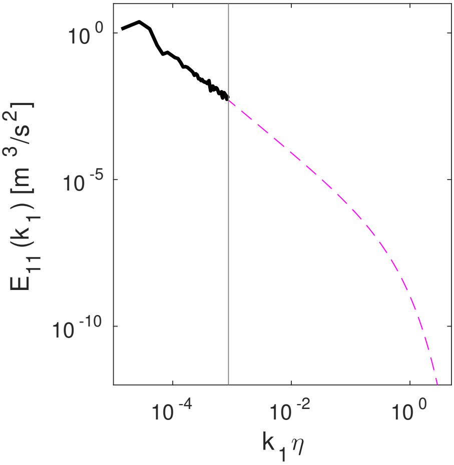
One-dimensional spectra E_{ij} - twice 1D Fourier transform of R_{ij}

$$E_{ij}(\kappa_l) = 2\mathcal{F}\left(R_{ij}\right)$$

In the inertial subrange

$$E_{LL}(\kappa_l) = C_k \epsilon^{2/3} \kappa_l^{-5/3}$$

Energy spectra



Energy spectra

Derivation of the energy spectrum function $E(\kappa)$.
Velocity spectrum tensor

$$\Phi_{ij}(\boldsymbol{x}, \boldsymbol{\kappa}, t) = \left(\frac{1}{2\Pi}\right)^3 \iiint R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\boldsymbol{r}} d\boldsymbol{r}.$$

Energy spectrum function is defined as

$$E(\kappa) = \frac{1}{2} \oint \Phi_{ii}(\boldsymbol{\kappa}, t) d\mathcal{S}$$

or, equivalently

$$E(\kappa) = \frac{1}{2} \iiint \Phi_{ii}(\boldsymbol{\kappa}, t) \delta(|\boldsymbol{\kappa}| - \kappa) d\mathcal{V}$$

Energy spectra

Turbulence kinetic energy

$$k = \int_0^\infty E(\kappa) d\kappa$$

Turbulence kinetic energy dissipation rate

$$\epsilon = \int_0^\infty 2\nu\kappa^2 E(\kappa) d\kappa$$

Relation between $E(\kappa)$ and the $1D$ spectral functions

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left(\frac{1}{\kappa}\frac{dE_{LL}}{d\kappa}\right)$$

References

S. B. Pope (2000)
Turbulent Flows
Cambridge University Press

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