Planetary boundary layer
and atmospheric turbulence.

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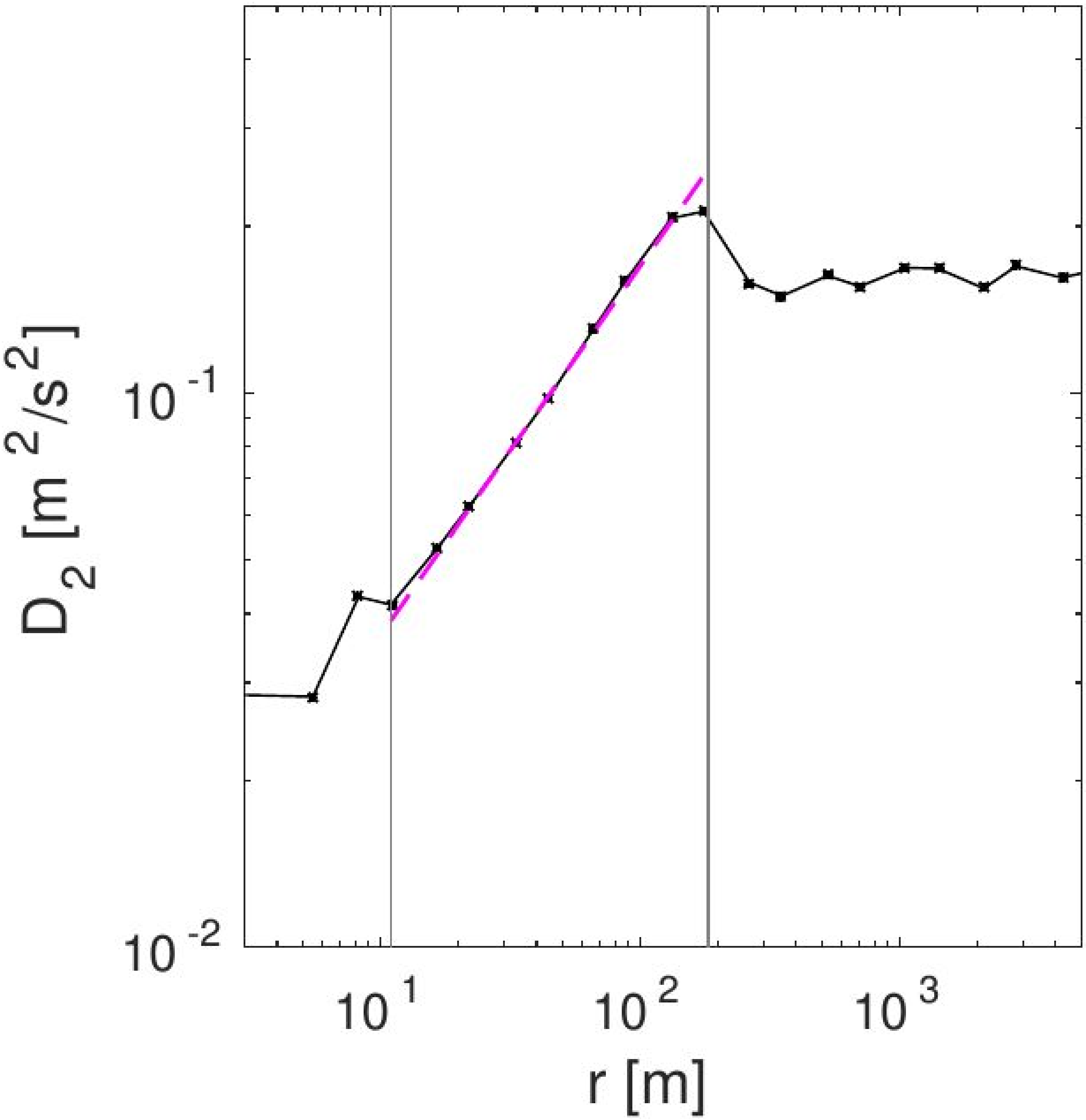
Lecture 04



Alternative approach: Fourier decomposition (no Frisch anymore).

For a flow which is homogeneous in space (i.e. statistical properties are independent of position), a spectral description is very appropriate, allowing us to examine properties as a function of wavelength. The total kinetic energy, given by

$$E = 1/2 \int u_i(\mathbf{x}) u_i(\mathbf{x}) d\mathbf{x} \tag{4.5}$$

can be written in terms of the spectrum $\phi_{i,j}(\mathbf{k})$

$$E = \frac{1}{2} \int \phi_{i,i}(\mathbf{k}) d\mathbf{k} = \int E(\mathbf{k}) d\mathbf{k} \tag{4.6}$$

where $\phi_{i,j}(\mathbf{k})$ is the Fourier transform of the velocity correlation tensor $R_{i,j}(\mathbf{r})$:

$$\phi_{i,j}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int exp(-i\mathbf{k}.\mathbf{r}) R_{i,j}(\mathbf{r}) d\mathbf{r} \; ; \; R_{i,j}(\mathbf{r}) = \int u_j(\mathbf{x}) u_i(\mathbf{x}+\mathbf{r}) d\mathbf{x} \tag{4.7}$$

$R_{i,j}(\mathbf{r})$ tells us how velocities at points separated by a vector $\mathbf{r}$ are related. If we know these two point velocity correlations, we can deduce $E(\mathbf{k})$. Hence the energy spectrum has the information content of the two-point correlation.

Notice that in 4.7 there are velocities in points x and x+r, which is similar to the 2nd order structure finction. In this equation, al well as in 4.5 there is velocity in second power!!!!i.

$E(\mathbf{k})$ contains directional information. More usually, we want to know the energy at a particular scale $k = \sqrt{\mathbf{k}.\mathbf{k}}$ without any interest in separating it by direction. To find $E(k)$, we integrate over the spherical shell of radius k (in 3-dimensions):

$$E = \int E(\mathbf{k}) d\mathbf{k} = \int_0^\infty \oint E(\mathbf{k}) d\sigma dk = \int_0^\infty E(k) dk \tag{4.8}$$

Then

$$E(k) = \oint E(\mathbf{k}) d\sigma = \frac{1}{2} \oint \phi_{i,i}(\mathbf{k}) d\sigma \tag{4.9}$$

Assuming isotropy:

$$E(k) = 2\pi k^2 \phi_{i,i}(k) \tag{4.10}$$

where $\phi_{i,i}(\mathbf{k}) = \phi_{i,i}(k)$ for all $\mathbf{k}$ such that $\sqrt{\mathbf{k}.\mathbf{k}} = k$.

Balance of energy in phase space.

We have an equation for the evolution of the total kinetic energy E. Equally interesting is the evolution of $E(k)$, the energy at a particular wavenumber k. This will include terms which describe the transfer of energy from one scale to another, via nonlinear interactions.

To obtain such an equation we first take the Fourier transform of the non-rotating, unstratified Boussinesq equations, using the following information about Fourier transforms:

Physical space	Fourier space
$f_i(\mathbf{x},t)$	$\hat{f}_i(\mathbf{k},t)$
$\partial f/\partial x_i$	$ik_i\hat{f}$
∇f	$i\hat{f}\mathbf{k}$
$\nabla^2 f$	$-k^2\hat{f}$
$f(\mathbf{x},t)g(\mathbf{x},t)$	$[\hat{f}*\hat{g}]$

where $[\hat{f}*\hat{g}] = \int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{f}(\mathbf{p},t)\hat{g}(\mathbf{q},t)d\mathbf{p}$

Then the momentum equation in physical space

$$\frac{\partial u_i}{\partial t} - \nu\frac{\partial^2 u_i}{\partial x_j^2} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j} \tag{4.11}$$

becomes, in fourier space:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i(\mathbf{k},t) = -ik_m\left(\delta_{i,j} - \frac{k_i k_j}{k^2}\right)\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_j(\mathbf{p},t)\hat{u}_m(\mathbf{q},t)d\mathbf{q} \tag{4.12}$$

(The term on the right hand side is the projection of the Fourier transform of $\mathbf{u}.\nabla\mathbf{u}$ onto the plane perpendicular to $\mathbf{k}$. The F.T. of ∇P is parallel to $\mathbf{k}$, while $\hat{\mathbf{u}}$ etc are all perpendicular to $\mathbf{k}$.)

The term on the right hand side shows that the nonlinear terms involve triad interactions between wave vectors such that $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

Now to obtain the energy equation we multiply eqn 4.12 by $\hat{u}_i(\mathbf{k}', t)$, similarly write an equation for $\hat{u}_i(\mathbf{k}', t)$ and multiply it by $\hat{u}_i(\mathbf{k}, t)$, and add the two equations together, and integrate over $\mathbf{k}'$ to obtain

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) \phi_{i,i}(\mathbf{k}, t) = \text{ Triad interaction terms} \tag{4.13}$$

Making use of eqn 4.10 (i.e. assuming isotropy), we then have

$$\frac{\partial}{\partial t} E(k, t) = T(k, t) - 2\nu k^2 E(k, t) \tag{4.14}$$

where $T(k, t)$ comprises the triad interaction terms. If we examine the integral of this equation over all k

$$\frac{\partial}{\partial t} \int_0^\infty E(k)dk = \int_0^\infty T(k, t)dk - 2\nu \int_0^\infty k^2 E(k)dk \tag{4.15}$$

and note that $-2\nu k^2 E(k)$ is the Fourier transform of the dissipation term $-\nu \nabla \mathbf{u}.\nabla \mathbf{u}$, then we see the familiar equation for the total energy budget eqn 4.2 is recovered only if

$$\int_0^\infty T(k, t)dk = 0 \tag{4.16}$$

Hence the nonlinear interactions transfer energy between different wave numbers, but do not change the total energy.

Now, adding a forcing term to the energy equation in k-space we have the following equation for energy at a particular wavenumber k:

$$\frac{\partial}{\partial t}E(k,t) = T(k,t) + F(k,t) - 2\upsilon k^2 E(k,t) \tag{4.17}$$

where $F(k,t)$ is the forcing term, and $T(k,t)$ is the **kinetic energy transfer**, due to nonlinear interactions. The **kinetic energy flux** through wave number k is $\Pi(k,t)$, defined as

$$\Pi(k,t) = \int_k^\infty T(k',t)dk' \tag{4.18}$$

or

$$T(k,t) = -\frac{\partial \Pi(k,t)}{\partial k} \tag{4.19}$$

Now for stationary turbulence

$$2\nu k^2 E(k) = T(k) + F(k) \tag{4.20}$$

If $F(k)$, the forcing, is concentrated on a narrow spectral band centered around a wave number k_i, then for $k \neq k_i$,

$$2\nu k^2 E(k) = T(k) \tag{4.21}$$

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In the limit of $\nu \to 0$, $T(k) = 0$. If the dissipation rate

$$\epsilon = \int_0^\infty 2\nu k^2 E(k) dk \tag{4.22}$$

then

$$\epsilon = \int_0^\infty F(k) dk \tag{4.23}$$

so that **the rate of dissipation of energy is equal to the rate of injection of energy**. Now in the limit of $\nu \to 0$, but nonzero $F(k)$, ϵ must remain nonzero, in order to balance the energy injection. (This is achieved by $\int_0^\infty k^2 E(k) dk \to \infty$). Then we find the energy flux in the limit $\nu \to 0$:

$$\Pi(k) = 0, \ : \ k < k_i$$
$$\Pi(k) = \epsilon \ : \ k > k_i \tag{4.24}$$

Hence at vanishing viscosity, the kinetic energy flux is constant and equal to the injection rate, for wavenumbers greater than the injection wavenumber k_i. Hence we have the following scenario: Energy is input at a rate ϵ at a wavenumber k_i, is fluxed to higher wavenumbers at a rate ϵ, and eventually dissipated at very high wavenumbers at a rate ϵ, even in the limit of $\nu \to 0$.

Kolmogorov's 1941 theory for the energy spectrum makes use of the result that ϵ, the energy injection rate, and dissipation rate also controls the flux of energy. Energy flux is independent of wavenumber k, and equal to ϵ for $k > k_i$. Kolmogorov's theory assumes the injection wavenumber is much less than the dissipation wavenumber ($k_i << k_d$, or large Re). In the intermediate range of scales $k_i < k < k_d$ neither the forcing nor the viscosity are explicitly important, but instead the energy flux ϵ and the local wavenumber k are the only controlling parameters. Then we can express the energy density as

$$E(k) = f(\epsilon, k) \tag{4.25}$$

Now using dimensional analysis:

Quantity	Dimension	
Wavenumber k	$1/L$	
Energy per unit mass E	$U^2 \sim L^2/T^2$	we find
Energy spectrum $E(k)$	$EL \sim L^3/T^2$	
Energy flux ϵ	$E/T \sim L^2/T^3$	

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \tag{4.26}$$

C_K is a universal constant known as the Kolmogorov constant. The region of parameter space in k where the energy spectrum follows this $k^{-5/3}$ form is known as the **Inertial range**. In this range, energy **cascades** from the larger scales where it was injected ultimately to the dissipation scale. The theory assumes that the spectra at any particular k depends only on spectrally local quantities - i.e. has no dependence on k_i for example. Hence the possibility for long-range interactions is ignored.

We can also derive the Kolmogorov spectrum in the following manner (after Obukhov): Define an eddy turnover time $\tau(k)$ at wavenumber k as the time taken for a parcel with energy $E(k)$ to move a distance $1/k$. If $\tau(k)$ depends only on $E(k)$ and k then, from dimensional analysis

$$\tau(k) \sim \left[k^3 E(k)\right]^{-1/2} \tag{4.27}$$

The energy flux can be defined as the available energy divided by the characteristic time τ. The available energy at a wavenumber k is of the order of $kE(k)$. Then we have

$$\epsilon \sim \frac{kE(k)}{\tau(k)} \sim k^{5/2} E(k)^{3/2} \tag{4.28}$$

and hence

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \tag{4.29}$$

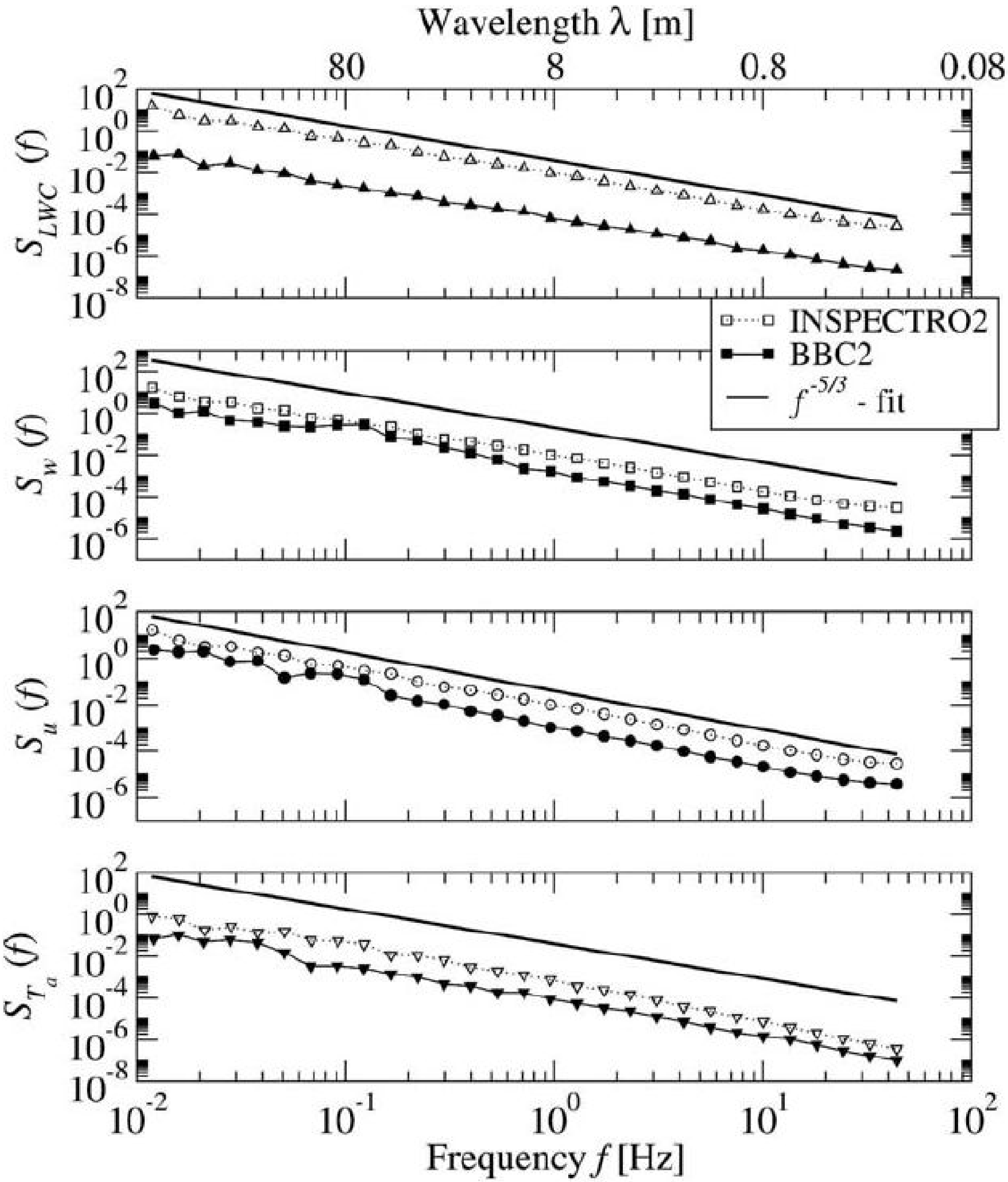
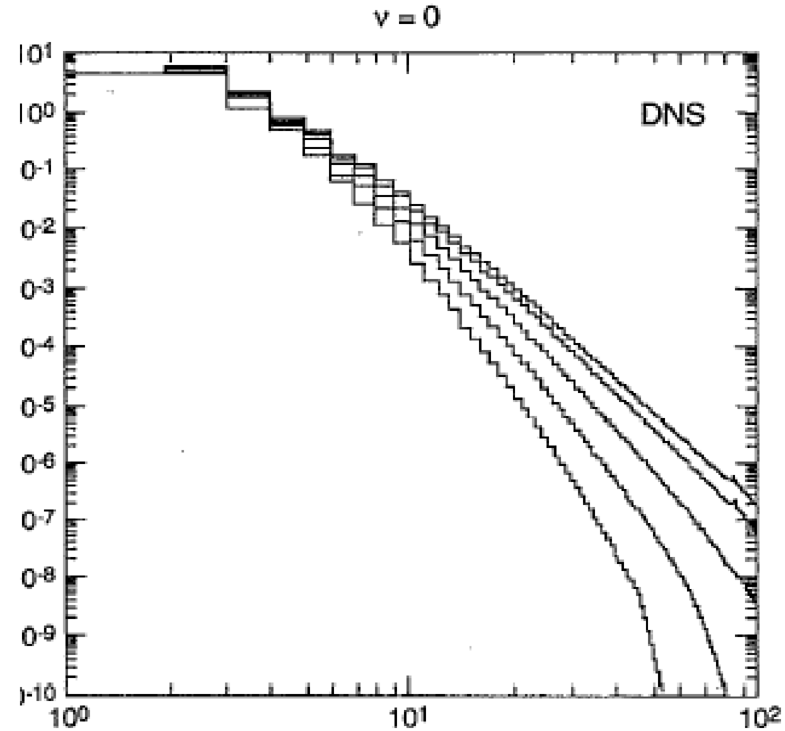


FIG. 5. Power spectral densities $S(f)$ of the same data as presented in Figs. 3 and 4. All spectra are in units of their variance per frequency; spectra of BBC data are divided by a factor of 10 for better resolution. For the top panel the frequencies are converted into wavelength assuming a constant horizontal wind speed of 8 m s^{-1}.



. DNS $E(k,t)$ for $t = (0.150, 0.175, 0.200, 0.225, 0.250)$, for initial
on (7), $\nu = 0$.

Kolmogorov scale and other characteristic scales of turbulence

Above a certain wavenumber k_d, viscosity will become important, and $E(k)$ will decay more rapidly than in the inertial range. The regime $k > k_d$ is known as the **dissipation range**. An estimate for k_d can be made by assuming

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} : k_i < k < k_d$$
$$E(k) = 0 : k > k_d \tag{4.30}$$

and substituting in eqn 4.22, and integrating between k_i and k_d. Then we have

$$k_d \sim \left(\frac{\epsilon^{1/4}}{\nu^{3/4}} \right) \tag{4.31}$$

The inverse $l_d = 1/k_d$ is known as the **Kolmogorov scale**, the scale at which dissipation becomes important.

$$l_d \sim \left(\frac{\nu^{3/4}}{\epsilon^{1/4}} \right) \tag{4.32}$$

Komogorov scale is often denoted as η

At the other end of the spectrum, the important lengthscale is l_i, the integral scale, the scale of the energy-containing eddies. $l_i = 1/k_i$. We can also evaluate l_i in terms of ϵ. We can write

$$\overline{u^2} = U^2 = \int_0^\infty E(k)dk \tag{4.33}$$

and substituting for $E(k)$ from eqn 4.26

$$U^2 = \int_0^\infty C_K \epsilon^{2/3} k^{-5/3} dk \tag{4.34}$$

Assume that 1/2 of the energy is contained at scales $k > k_i$. Then

$$U^2 = 6C_K \epsilon^{2/3} k_i^{-2/3} \tag{4.35}$$

and

$$k_i \sim \frac{\epsilon}{U^3} \tag{4.36}$$

so that $l_i \sim U^3/\epsilon$. Then the ratio of maximum and minimum dynamically active scales

$$\frac{l_i}{l_d} = \frac{k_d}{k_i} \sim \frac{U^3}{\epsilon^{3/4}\nu^{3/4}} \sim \left(\frac{U l_i}{\nu}\right)^{3/4} \sim Re_{l_i}^{3/4} \tag{4.37}$$

where Re_{l_i} is the **Integral Reynolds number**. Hence the range of scales goes as the Reynolds number to the power 3/4. This information is useful in estimating numerical resolution necessary to simulate turbulence down to the Kolmogorov scale at a chosen Reynolds number.

Taylor microscale.

A third length scale often used to characterise turbulence is the **Taylor microscale**:

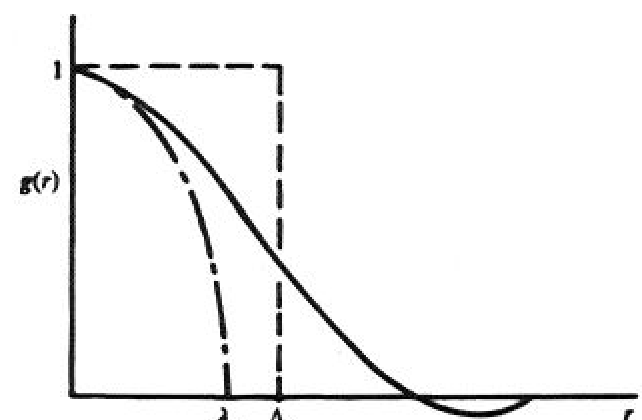
$$\lambda = \left(\frac{\overline{u_i^2}}{\overline{(\partial u_i / \partial x_j)^2}} \right)^{1/2} = \left(\frac{U^2 \nu}{\epsilon} \right)^{1/2} \quad (4.38)$$

The Taylor microscale is the characteristic spatial scale of the velocity gradients. Using λ, an alternative Reynolds number can be defined:

$$Re_\lambda = \frac{U\lambda}{\nu} = \frac{U^2}{\nu^{1/2}\epsilon^{1/2}} \quad (4.39)$$

where $Re_\lambda \sim Re_{l_i}^{1/2} \sim l_i/\lambda$.

Taylor microscale Reynolds number



Turbulent Kinetic Energy, second order moments – phenomenology

Adding the contributions due to the 3 velocity components and rewriting in Einstein notation we have

$$\left(\frac{\partial}{\partial t} + \overline{U}_j \frac{\partial}{\partial x_j}\right) \frac{\overline{u_i'^2}}{2} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho_0}\overline{u_i' p'}\delta_{i,j} + \nu \frac{\partial}{\partial x_j}\frac{\overline{u_i'^2}}{2} - \overline{u_j' u_i' u_i'}\right)$$

$$-\nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} - \overline{u_j' u_i'}\frac{\partial}{\partial x_j}\overline{U}_i + \overline{b'w'} \qquad (3.14)$$

Transport

Hence TKE is generated by (a) shear production,

$$P = -\overline{u_j' u_i'}\frac{\partial}{\partial x_j}\overline{U}_i \qquad (3.15)$$

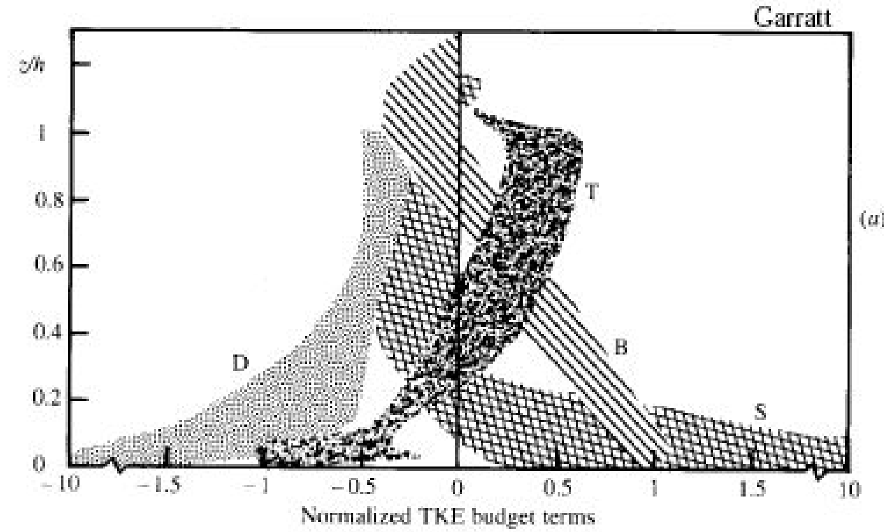
and (b) buoyant production

$$B = \overline{b'w'} \qquad (3.16)$$

and lost through dissipation

$$\epsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \qquad (3.17)$$

The buoyant production term may be either positive (generation of kinetic energy, loss of potential energy) or negative (loss of KE, increase in PE).



Stull's textbook

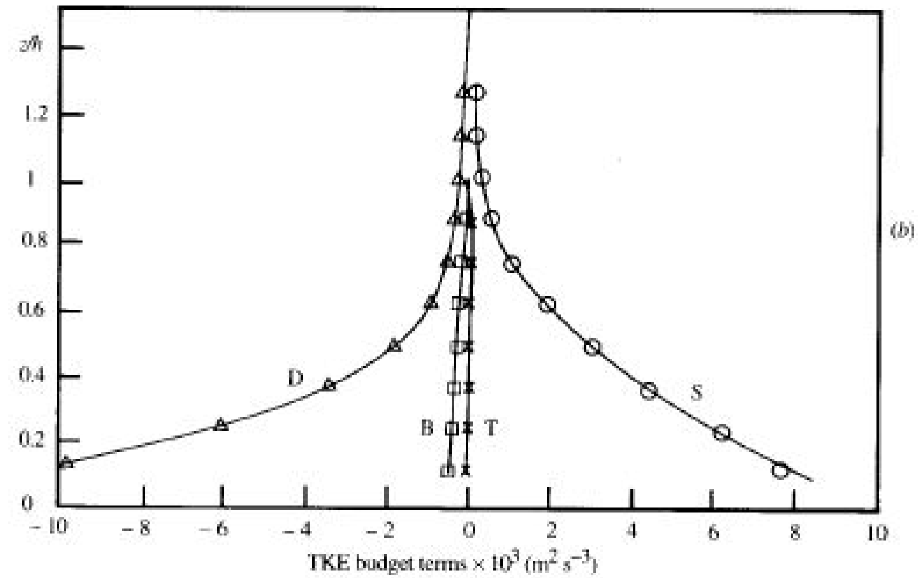


Fig. 2.4 Terms in the TKE equation (2.74b) as a function of height, normalized in the case of the clear daytime ABL (*a*) through division by w_*^3/h; actual terms are shown in (*b*) for the clear night-time ABL. Profiles in (*a*) are based on observations and model simulations as described in Stull (1988; Figure 5.4), and in (*b*) are from Lenschow *et al.* (1988) based on one aircraft flight. In both, B is the buoyancy term, D is dissipation, S is shear generation and T is the transport term. Reprinted by permission of Kluwer Academic Publishers.

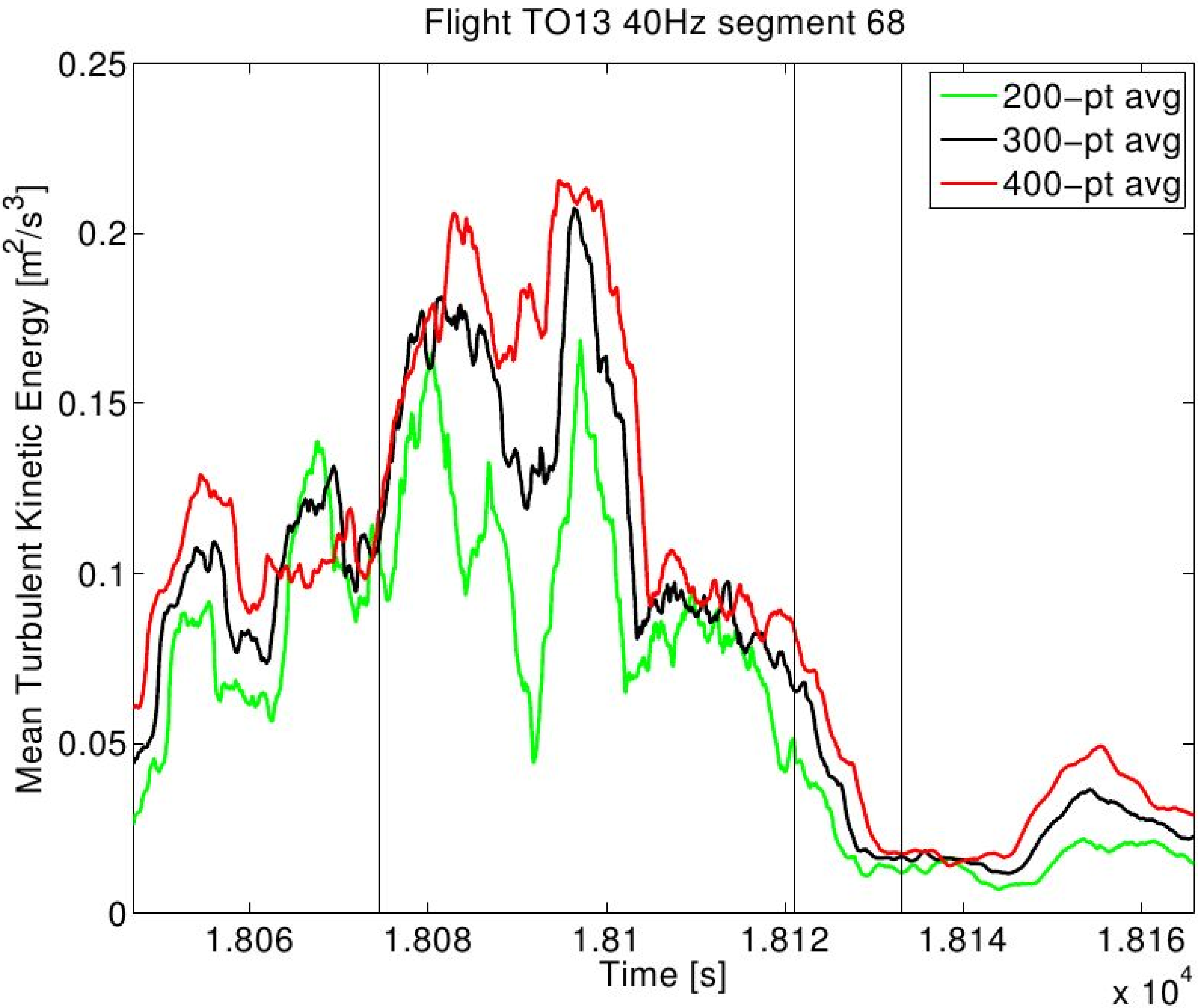
POST – Physics of Stratocumulus Top, California, 2008

aerosol (CCN)

microphysics

temperature,
humidity,
liquid water,
turbulence,

droplet counting



Airborne data in clouds: Estimates of TKE depend on averaging!

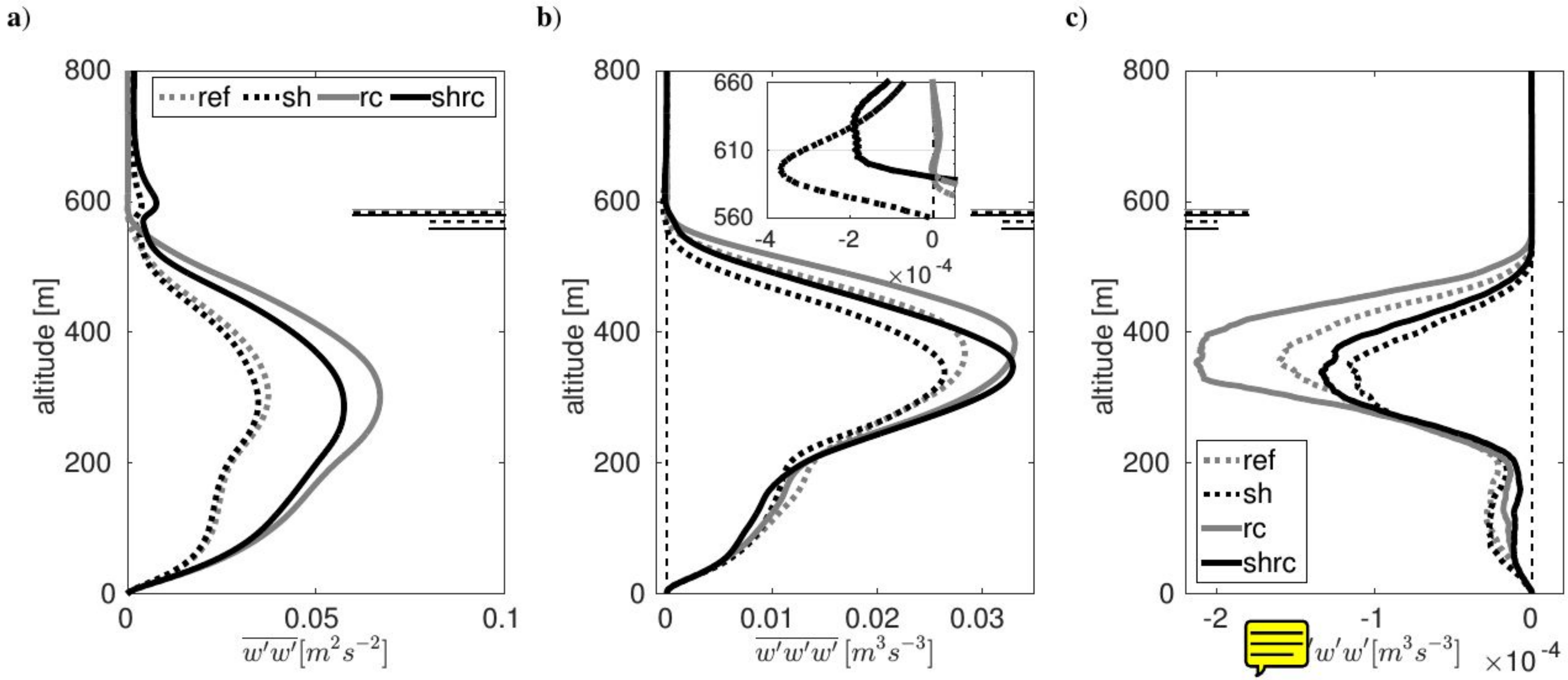


Figure 10. *a)* Vertical velocity variance. *b)* Vertical velocity 3rd moment. *c)* Median of $w'w'w'$. The colour code is as in Figure 4. Short horizontal lines indicate averaged cloud top (z_c) and long horizontal lines marks the averaged level of maximum gradient of liquid water potential temperature (z_i).

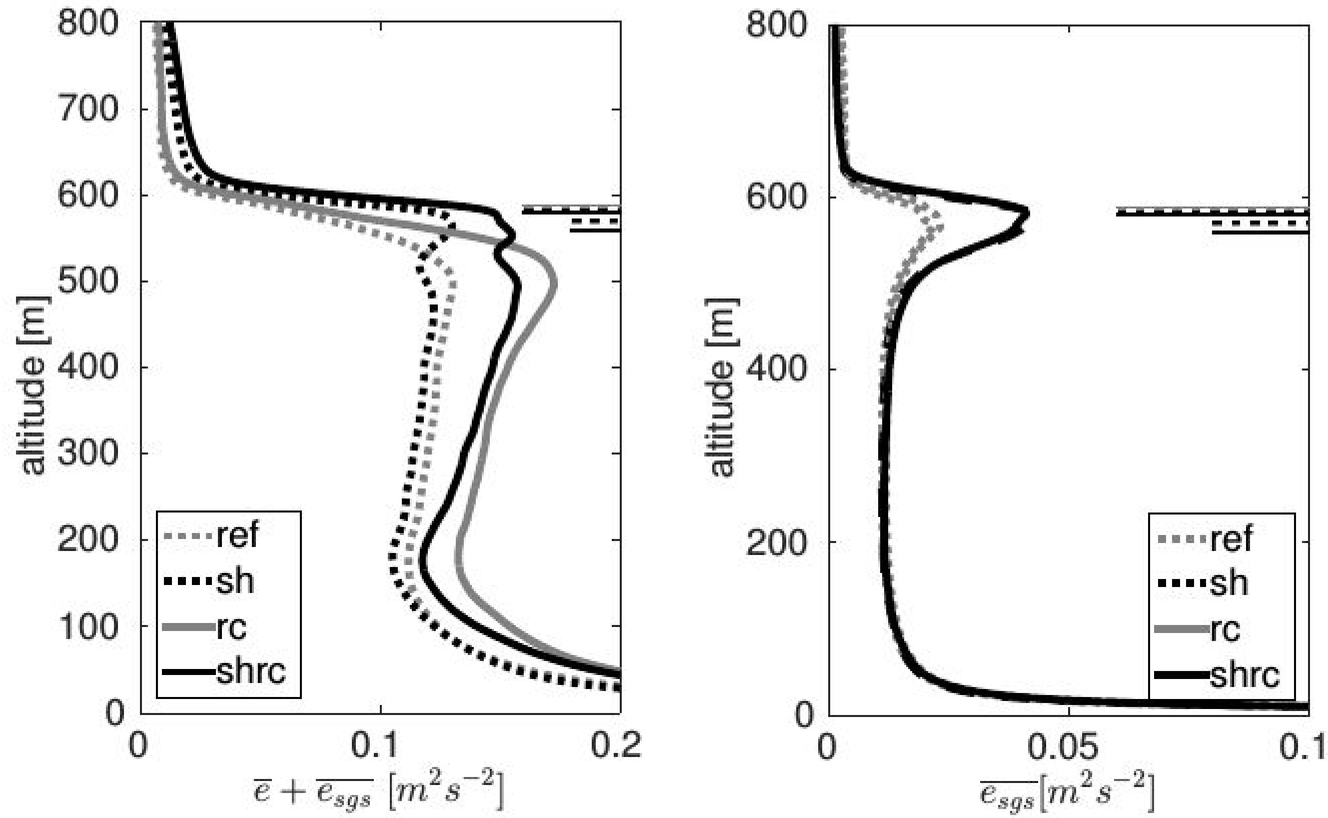


Figure 11. Resolved plus SGS TKE (left panel) and SGS TKE (right panel). The colour code is as in Figure 4. Short horizontal lines indicate cloud top and long horizontal lines marks the level of maximum gradient of liquid water potential temperature.

$$\frac{\partial \overline{e}}{\partial t} + \overline{w}\frac{\partial \overline{e}}{\partial z} = \frac{g}{\overline{\theta_v}}\overline{w'\theta_v'} - \left(\overline{u'w'}\frac{\partial \overline{u}}{\partial z} + \overline{v'w'}\frac{\partial \overline{v}}{\partial z}\right) - \frac{\partial \overline{w'e}}{\partial z} + \\ - \frac{1}{\overline{\rho}}\frac{\partial \overline{w'p'}}{\partial z} - \epsilon, \quad (3)$$

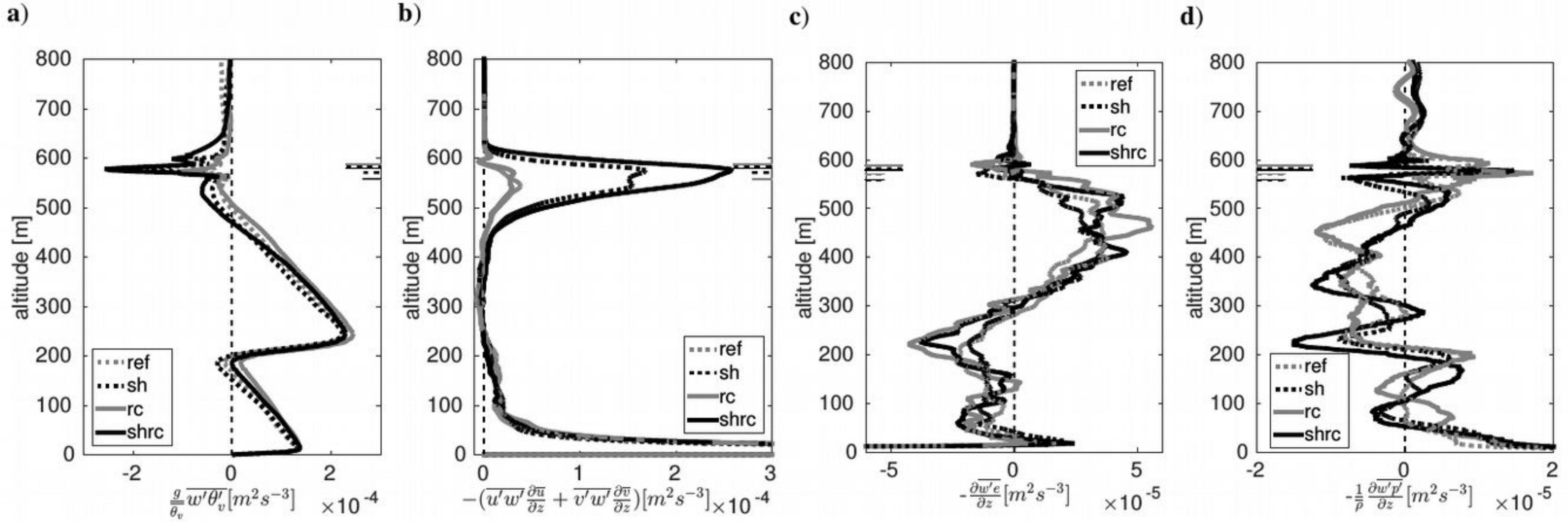


Figure 12. Time-averaged TKE budget terms calculated for the last hour of the simulations. Colour code as in Figure 4. Short horizontal lines indicate cloud top and long horizontal lines marks the level of maximum gradient of liquid water potential temperature.

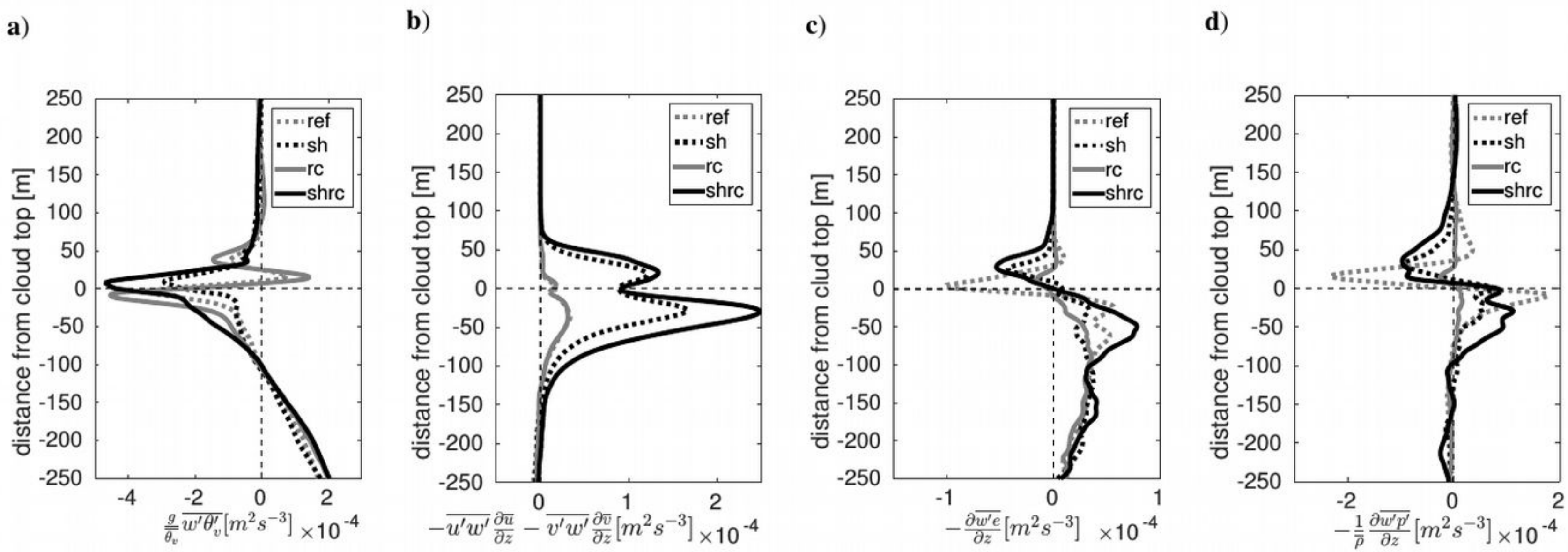
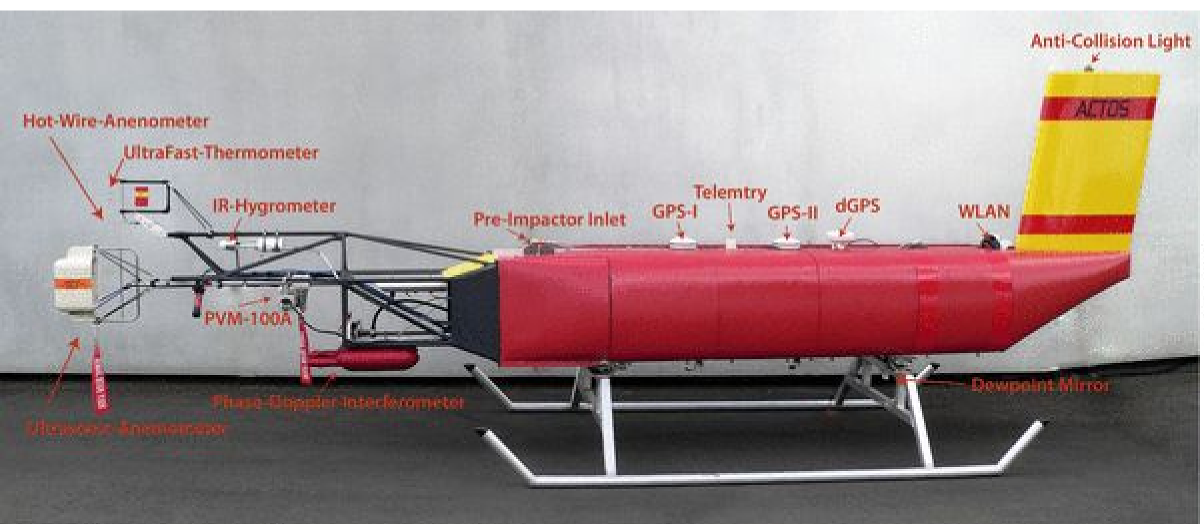


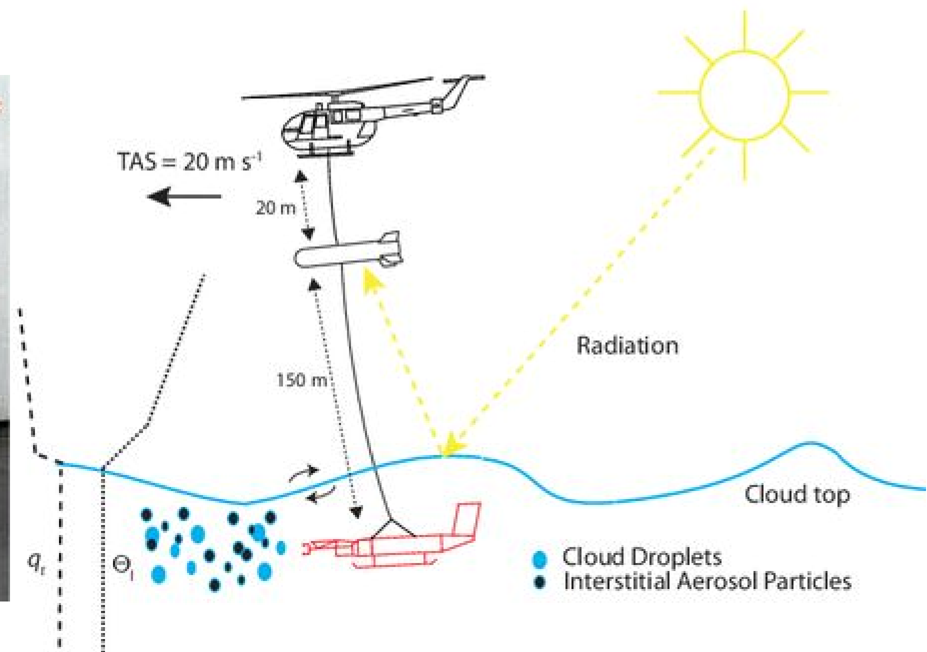
Figure 13. Time-averaged TKE budget terms calculated for the last hour of the simulations and normalised to the cloud top. Colour code as in Figure 4.

Airborne

Cloud Turbulence Observation System

ACTOS





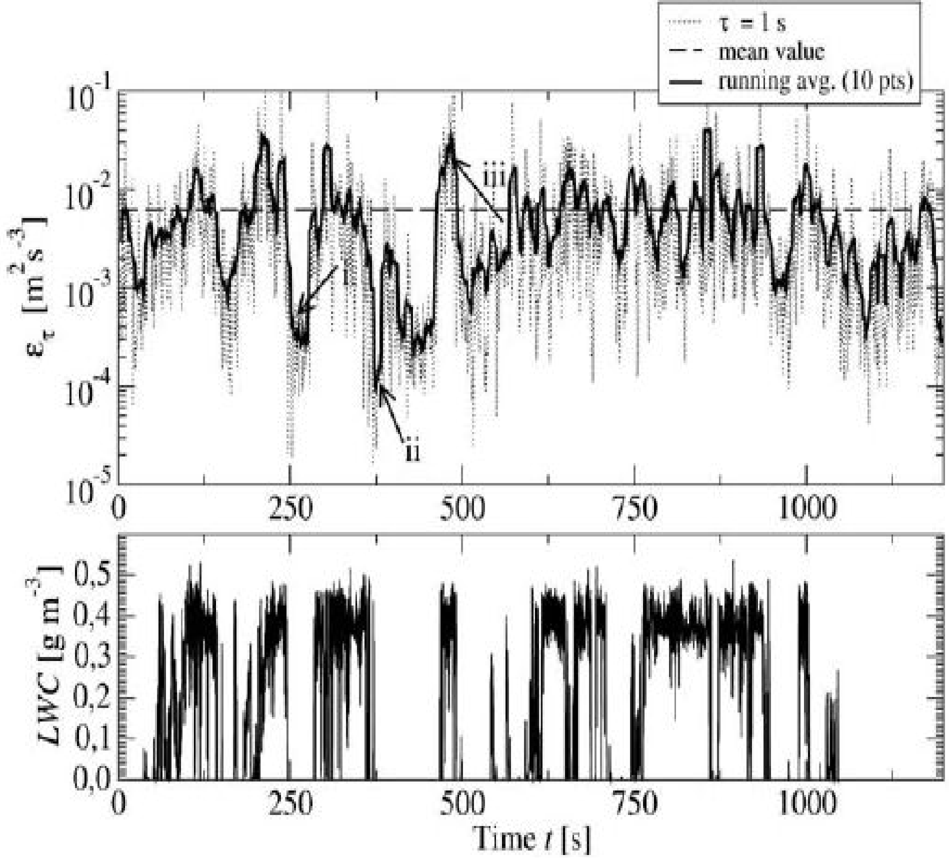


FIG. 8. (top) Time series of local energy dissipation rate ε_τ and (bottom) LWC of BBC2 data. The integration time τ for ε_τ is 1 s; a running average over 10 points is included.

TKE dissipation ate estimated from the balloon-borne measurements.

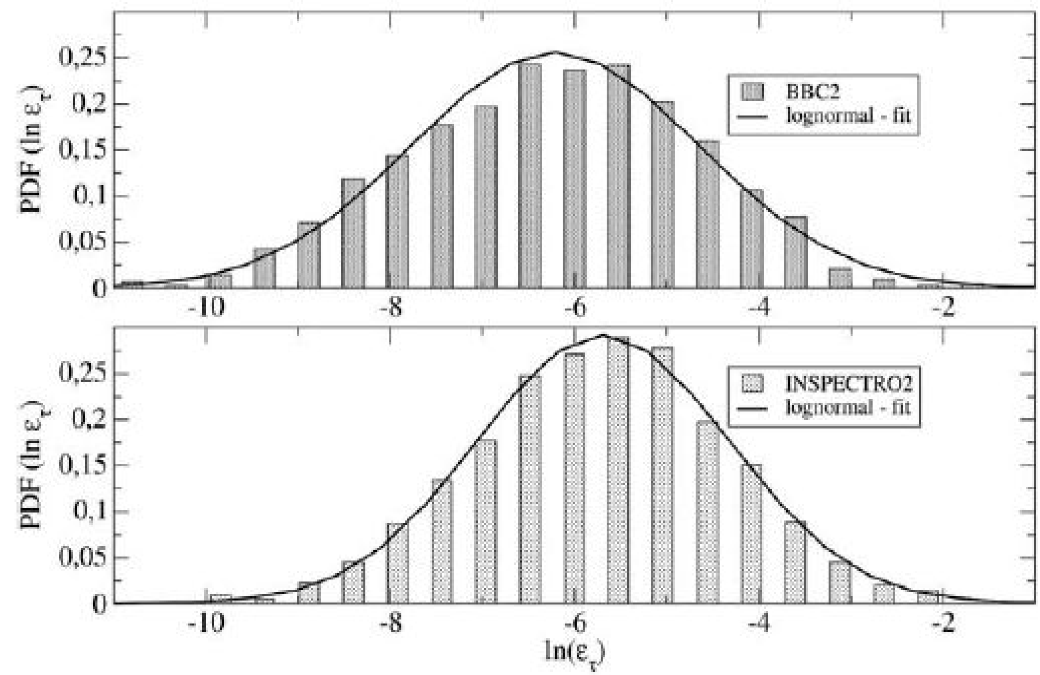


FIG. 11. PDF of natural logarithm of local energy dissipation rates ε_τ. A Gauss fit is included for reference.