Turbulence and atmospheric boundary layer
Lecture 9

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Summary of lecture 8

1. Phenomenology of the atmospheric boundary layer (ABL)
2. Structure of the ABL
3. Diurnal cycle
4. Dimensional analysis
5. Transport equations in the boundary layer approximation

BL approximation

Momentum balance - u component

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + \nu\frac{\partial^2 \overline{u}}{\partial z^2} + f\overline{v}$$

Momentum balance - v component

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} + \frac{\partial \overline{v'w'}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial y} + \nu\frac{\partial^2 \overline{v}}{\partial z^2} - f\overline{u}$$

Momentum balance - vertical w component

$$\frac{\partial \overline{w'^2}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \overline{b}$$

BL approximation

Continuity

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

Energy balance

$$\frac{\partial \overline{\theta}}{\partial t} + \overline{u}\frac{\partial \overline{\theta}}{\partial x} + \overline{v}\frac{\partial \overline{\theta}}{\partial y} + \overline{w}\frac{\partial \overline{\theta}}{\partial z} + \frac{\partial \overline{\theta' w'}}{\partial z} = \kappa \frac{\partial^2 \overline{\theta}}{\partial z^2} + S_\theta$$

BL approximation

Balance of mean mixing ratio

$$\frac{\partial \overline{q}}{\partial t} + \overline{u}\frac{\partial \overline{q}}{\partial x} + \overline{v}\frac{\partial \overline{q}}{\partial y} + \overline{w}\frac{\partial \overline{q}}{\partial z} + \frac{\partial \overline{q'w'}}{\partial z} = \kappa \frac{\partial^2 \overline{q}}{\partial z^2} + S_q$$

The effect of turbulence on the mean profiles $\overline{u}$, $\overline{\theta}$ and $\overline{q}$ is felt through the derivative of turbulent fluxes $\overline{u'w'}$, $\overline{\theta' w'}$, $\overline{q'w'}$, respectively, in the vertical direction.

$\rho_0 C_p \overline{\theta' w'}$ - turbulent sensible heat flux,
$\rho_0 \mathcal{L} \overline{q'w'}$ - turbulent latent heat flux.

Mean winds and temperature profiles in the surface layer

Mean winds and temperature profiles in the surface layer

The balance equations cannot be solved exactly due to the presence of unclosed turbulent flux terms.
Universal profiles of the mean temperature and mean winds were described with the use of dimensional analysis by Russian scientists A. S. Monin and A. M. Obukhov.

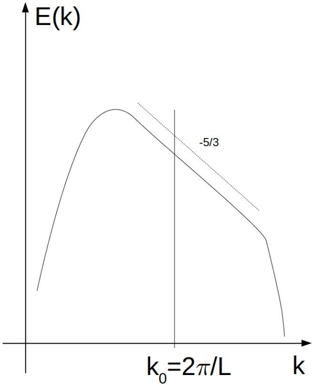
Results are known as the Monin-Obuhkov similarity theory.

To start with let us recall the basic assumptions of the gradient diffusion hypothesis and the Prandtl mixing-length hypothesis for neutrally stratified ABL.

Gradient diffusion hypothesis

In RANS turbulence models eddy viscosity and eddy diffusivity were introduced to replace the whole range of scales in turbulence by a single, characteristic scale L with the corresponding velocity scale U.

1. Turbulent (eddy) viscosity $\nu_t \sim UL$
2. Turbulent (eddy) diffusivity $\kappa_t = \nu_t / Pr_t$



$Pr_t = \mathcal{O}(1)$ is the turbulent Prandtl number

Gradient diffusion hypothesis

Reynolds stresses

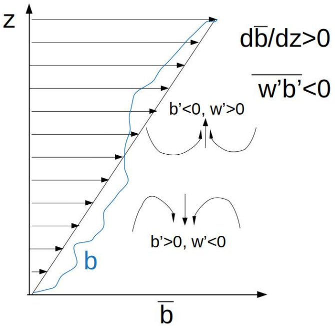
$$\overline{u'w'} = -\nu_t \frac{\partial \overline{u}}{\partial z}$$

$$\overline{v'w'} = -\nu_t \frac{\partial \overline{v}}{\partial z}$$

Turbulent fluxes

$$\overline{b'w'} = -\kappa_t \frac{\partial \overline{b}}{\partial z}$$

$$\overline{q'w'} = -\kappa_t \frac{\partial \overline{q}}{\partial z}$$



Gradient diffusion hypothesis

0-equation models

$\nu_t \sim UL$

where

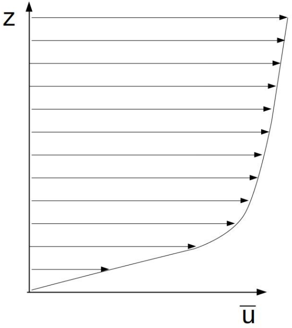
$$L \sim l_m, \quad U \sim l_m \left| \frac{\partial \overline{u}}{\partial z} \right|$$

l_m is the mixing length

Near the wall, in the so-called log-law region

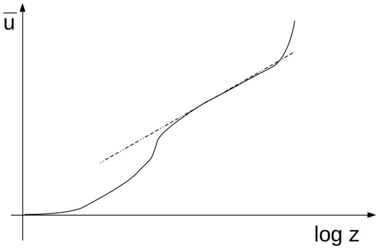
$$l_m \sim z$$

$$\nu_t \sim z^2 \left| \frac{\partial \overline{u}}{\partial z} \right|$$



Logarithmic profile of mean velocity

In the momentum balance equations we assume stationarity, neglect mean convection terms (gradients in x and y are negligible and $\overline{w} = 0$), Coriolis force (this can be done in the surface layer), the viscous term (which is important only in the vicinity of the surface) and the mean pressure gradient (a wall placed in free external flow).



$$\frac{d\overline{u'w'}}{dz} = 0$$

Logarithmic profile of mean velocity

So, $\overline{u'w'} = const$.
Introducing the gradient diffusion model

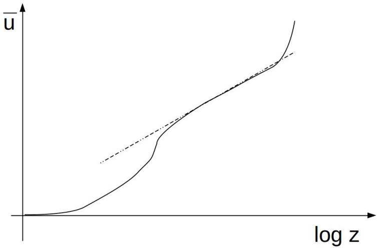
$$\nu_t \frac{d\overline{u}}{dz} = const$$

hence

$$z^2 \left| \frac{d\overline{u}}{dz} \right| \frac{d\overline{u}}{dz} = const$$

and it follows that

$$z\frac{d\overline{u}}{dz} = const, \quad \overline{u} \sim log\, z + const$$



Monin-Obukhov theory

In the case of neutral stratification

$$z\frac{d\overline{u}}{dz} = const, \quad z\frac{d\overline{\theta}}{dz} = const$$

Let us introduce velocity and temperature scales

$$u_\tau = \left(-\nu_t \frac{d\overline{u}}{dz}\Big|_{z=0}\right)^{1/2} = (\overline{u'w'})^{1/2}, \quad \theta_\tau = -\overline{\theta' w'}/u_\tau$$

We assume the two scales are sufficient to describe the phenomenon

$$\frac{\alpha z}{u_\tau}\frac{d\overline{u}}{dz} = 1, \quad \frac{z}{\theta_\tau}\frac{d\overline{\theta}}{dz} = 1$$

where $\alpha \approx 0.4$ is the Kolmogorov's constant.

Monin-Obukhov theory

Neutral stratification

$$\frac{\alpha z}{u_\tau}\frac{d\overline{u}}{dz} = 1, \qquad \frac{z}{\theta_\tau}\frac{d\overline{\theta}}{dz} = 1$$

Monin-Obukhov theory assumes in the non-neutral case the phenomenon is governed by the following parameters:

$$u_\tau, \; Q = -\rho c_p \theta_\tau u_\tau, \; g/T_0$$

where T_0 is the surface temperature, Q is the turbulent heat flux. Note that the gravity g comes into play in the non-neutral case.

Monin-Obukhov theory

Using the parameters

$$u_\tau, \; Q = -\rho c_p \theta_\tau u_\tau, \; g/T_0$$

we can construct one length scale, called the Monin-Obukhov length

$$L = -\frac{u_\tau^3}{\alpha \frac{g}{T_0} \frac{Q}{\rho c_p}}$$

When $L < 0$ the surface is statically unstable, when $L > 0$ it is statically stable. In the neutral case $L \to \infty$. So, the non-dimensional variable z/L

$$\text{unstable stratification:} \quad \frac{z}{L} < 0$$

$$\text{stable stratification:} \quad \frac{z}{L} > 0$$

$$\text{neutral stratification:} \quad \frac{z}{L} = 0$$

Monin-Obukhov theory

The Monin-Obukhov theory assumes the existence of universal functions $\phi_M(z/L)$ and $\phi_H(z/L)$

Non-neutral stratification

$$\frac{\alpha z}{u_\tau}\frac{d\overline{u}}{dz} = \phi_M\left(\frac{z}{L}\right), \quad \frac{z}{\theta_\tau}\frac{d\overline{\theta}}{dz} = \phi_H\left(\frac{z}{L}\right)$$

for small z/L, such that $|z/L| < 1$ the functions can be expanded in the Taylor-series

$$\phi_M\left(\frac{z}{L}\right) \approx 1 + 8.1\frac{z}{L}$$

In unstable case better results are obtained for

$$\phi_M\left(\frac{z}{L}\right) \approx 1 - 15\left(\frac{z}{L}\right)^{-1/4}$$

Monin-Obukhov theory

We can calculate solution for the mean wind for weak stratification:

$$\frac{\alpha z}{u_\tau}\frac{d\overline{u}}{dz} = \phi_M\left(\frac{z}{L}\right), \quad \phi_M\left(\frac{z}{L}\right) \approx 1 + \beta\frac{z}{L}$$

$$d\overline{u} = \frac{u_\tau}{\alpha}\left[\frac{1}{z} + \frac{\beta}{L}\right] dz$$

We solve it with the boundary condition $\overline{u} = 0$ at $z = z_0$ and obtain the log-linear profile

$$\overline{u} = \frac{u_\tau}{\alpha}\left(\ln\frac{z}{z_0} + \beta\frac{z - z_0}{L}\right)$$

Monin-Obukhov theory

Figure: Typical profiles of mean winds in the stable, neutral and stable cases a) linear plot, b) semi-log plot

picture from:
Meteorology for Scientists and Engineers, A Technical Companion Book to C. Donald Ahrens Meteorology Today, 2nd Ed., by Stull, p. 77

Monin-Obukhov theory

A lot of experimental efforts were devoted to finding empirical form of the functions $\phi_M(z/L)$ and $\phi_H(z/L)$ by fitting to experimental data. The most famous experiment was the Kansas measurement campaign, where consistency between measurements and predictions from similarity relations for the entire range of stability values was reported.

Businger, J. A.; J. C. Wyngaard; Y. Izumi; E. F. Bradley (1971). "Flux-profile relationships in the atmospheric surface layer". Journal of the Atmospheric Sciences. 28 (2): 181189.

Figure: Kansas field.

Photo by Chris Light, source: https://en.wikipedia.org/wiki/Monin%E2%80%93Obukhov_similarity

Monin-Obukhov theory

Different, improved empirical fits were proposed, e.g.

$$\begin{aligned}
\phi_M &= \left(1 - 19.3\frac{z}{L}\right)^{-1/4}, \quad -2 < \frac{z}{L} < 0 \\
\phi_M &= 1 + 6\frac{z}{L}, \quad 0 < \frac{z}{L} < 1 \\
\phi_H &= 0.95\left(1 - 11.6\frac{z}{L}\right)^{-1/4}, \quad -2 < \frac{z}{L} < 0 \\
\phi_H &= 0.95 + 7.8\frac{z}{L}, \quad 0 < \frac{z}{L} < 1
\end{aligned}$$

Foken T (2006) 50 Years of the Monin-Obukhov similarity theory. Boundary-Layer Meteorology. 2: 729.

Monin-Obukhov theory

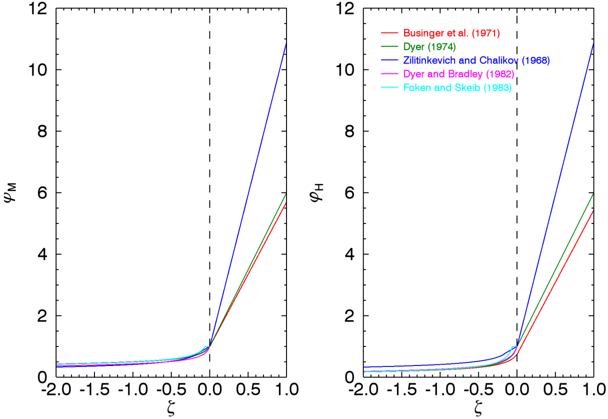


Figure: Non-dimensional functions $\phi_M(\xi)$, $\phi_H(\xi)$. Picture by Geoalchimista, source: https://en.wikipedia.org/wiki/Monin%E2%80%93Obukhov_similarity_theory, CC BY-SA 4.0

Mean winds in the outer layer

Outer layer

Geostrophic balance (stationarity, negligible convection, no turbulence) above the capping inversion:

$$
\begin{aligned}
0 &= -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\,v_g \\
0 &= -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial y} - f\,u_g
\end{aligned}
$$

To find profiles in the ABL, below the inversion, we can replace the pressure terms with formulas for the geostrophic wind:

$$
\begin{aligned}
\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} &= -\frac{\partial \overline{u'w'}}{\partial z} + f(\overline{v} - v_g) \\
\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} &= -\frac{\partial \overline{v'w'}}{\partial z} - f(\overline{u} - u_g)
\end{aligned}
$$

Fridtjof Nansen, a Norwegian explorer and scientist observed during his expedition in 1890s that the ice transport appeared to occur at an angle to the wind direction.
This phenomenon was later explained by the Swedish oceanographer, Vagn Walfrid Ekman (1902).

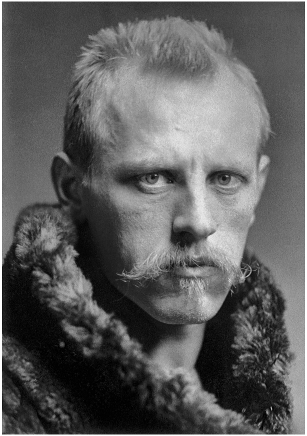


Figure: Norwegian scientist, diplomat and explorer Fridtjof Nansen, https://pl.wikipedia.org/wiki/Fridtjof_Nansen (public domain)

Ekman spiral

Let us assume, that within the outer layer the eddy viscosity and density are constant, the motion is horizontal and steady, the isobars are straight and parallel, and the geostrophic wind is constant with height. The x direction is taken along the pressure gradient.

Figure: Vagn Ekman, Swedish oceanographer, 1874-1954 (by unknown author, source: https://en.wikipedia.org/wiki/Vagn_Walfrid_Ekman)

Ekman spiral

With such assumptions the LHS of the momentum equations (time derivative + convection) are neglected and the turbulent fluxes can be approximated with the use of the eddy diffusivity hypothesis

$$\overline{u'w'} = -\nu_T \frac{\partial \overline{u}}{\partial z}$$

$$\overline{v'w'} = -\nu_T \frac{\partial \overline{v}}{\partial z}$$

where ν_T is the eddy diffusivity.

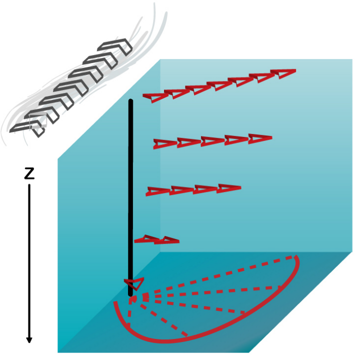


Figure: By Nedtheprotist (public domain)

https://commons.wikimedia.org/w/index.php?curid=5403984

Ekman spiral

With this we obtain

$$\nu_T \frac{d^2\overline{u}}{dz^2} = -f\overline{v}$$
$$\nu_T \frac{d^2\overline{v}}{dz^2} = f(\overline{u} - u_g)$$

This is a system of two ODE's - easy to solve with the boundary conditions

$$z = 0 : \quad \overline{u} = 0, \ \overline{v} = 0$$

$$z = \infty : \quad \overline{u} = G, \ \overline{v} = 0$$

where $G = \sqrt{(u_g^2 + v_G^2)}$

Ekman spiral

Solution:

$$\begin{aligned}\overline{u} &= G\left(1-\mathrm{e}^{-z/d}\cos\frac{z}{d}\right)\\ \overline{v} &= G\,\mathrm{e}^{-z/d}\sin\frac{z}{d}\end{aligned}$$

where

$$d=\left(\frac{2\nu_T}{f}\right)^{1/2}$$

is called the Ekman depth.
It is seen, the wind changes its magnitude and turns with the height.

Ekman spiral

This behavior can be presented on the velocity hodograph.

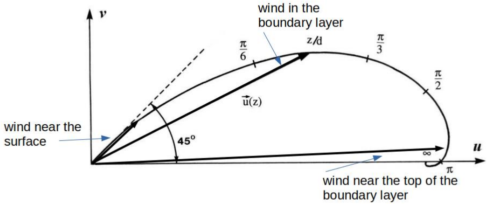


Figure: The velocity spiral in the bottom Ekman layer (for Northen hemisphere where $f > 0$).

Picture replotted based on a figure 8-4 from: Introduction to Geophysical Fluid Dynamics by B. Cushman-Roisin J.-M. Beckers

Budget of turbulence kinetic energy in ABL

k transport equation

$$\frac{\partial k}{\partial t} + \overline{u}\frac{\partial k}{\partial x} + \overline{v}\frac{\partial k}{\partial y} + \overline{w}\frac{\partial k}{\partial z} = \underbrace{\overline{w'b'}}_{B} - \underbrace{\frac{\partial}{\partial z}\left(\frac{1}{2}\overline{u_i'u_i'w'} + \frac{1}{\rho_0}\overline{p'w'}\right)}_{T}$$

$$- \underbrace{\overline{u'w'}\frac{\partial \overline{u}}{\partial z} + \overline{v'w'}\frac{\partial \overline{v}}{\partial z}}_{\mathcal{P}} + \underbrace{\nu\frac{\partial^2 k}{\partial z^2}}_{D} - \epsilon$$

where B is the buoyancy, T is the transport, $\mathcal{P}$ is the production, D is the molecular transport (valid only in the vicinity of the surface), ϵ is the dissipation rate

Budget of turbulence kinetic energy in ABL

Figure: TKE budget, left: unstable, right: stable
picture from: http://climate-dynamics.org/wp-content/uploads/2015/05/Lecture_TKE_150310.pdf

based on: An Introduction to Boundary Layer Meteorology, R. B. Stull (2002)

References

J.R. Garratt (1992)
The atmospheric boundary layer
Cambridge University Press

R. Stull (2005)
The atmospheric boundary layer In: Atmospheric Science (Eds. J. Wallace,P. V. Hobbs)
Elsevier

The End